TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 249, Number 2, May 1979

EXTENSIONS, RESTRICTIONS, AND REPRESENTATIONS OF STATES ON C^*-ALGEBRAS

BY
JOEL ANDERSON[1]

ABSTRACT. In the first three sections the question of when a pure state g on a C^*-subalgebra B of a C^*-algebra A has a unique state extension is studied. It is shown that an extension f is unique if and only if $\inf\|b(a - f(a)1)b\| = 0$ for each a in A, where the inf is taken over those b in B such that $0 \leq b \leq 1$ and $g(b) = 1$. The special cases where B is maximal abelian and/or $A = B(H)$ are treated in more detail. In the remaining sections states of the form $T \mapsto \lim_{\mathfrak{U}}(Tx_\alpha, x_\alpha)$, where $\{x_\alpha\}_{\alpha\in\kappa}$ is a set of unit vectors in H and $\mathfrak{U}$ is an ultrafilter are studied.

Introduction. In **[10]** Kadison and Singer studied the question: if $\mathfrak{B}$ is a maximal abelian subalgebra of $\mathfrak{B}(\mathfrak{H})$ (the set of bounded linear operators on a separable Hilbert space) does each homomorphism of $\mathfrak{B}$ have a unique state extension to $\mathfrak{B}(\mathfrak{H})$? They showed that if $\mathfrak{B}$ is isomorphic to $L^\infty(0, 1)$ then there are homomorphisms of $\mathfrak{B}$ for which distinct state extensions exist. More recently in **[13]** Reid showed that if $\mathfrak{B}$ is a maximal abelian subalgebra of $\mathfrak{B}(\mathfrak{H})$ which is isomorphic to $l^\infty(\mathbf{N})$, where $\mathbf{N}$ denotes the positive integers, then there are (nontrivial) homomorphisms of $\mathfrak{B}$ which do have unique state extensions to $\mathfrak{B}(\mathfrak{H})$.

In §§1–3 of this paper we study the extension question for arbitrary C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and give generalizations of some of the results in **[10]**. For example in §3 we show that if $\mathfrak{B} \subset \mathfrak{A}$ and f is a pure state on $\mathfrak{B}$, then f has a unique state extension to $\mathfrak{A}$ if and only if for each T in $\mathfrak{A}$ there is a positive element B in $\mathfrak{B}$ so that $f(B) = \|B\| = 1$ and the distance from BTB to $\mathfrak{B}$ is arbitrarily small. Using this fact we show that if $\mathfrak{B}$ is a maximal abelian subalgebra of $\mathfrak{A}$ then each nonzero homomorphism of $\mathfrak{B}$ has a unique state extension to $\mathfrak{A}$ if and only if there is a conditional expectation of $\mathfrak{A}$ onto $\mathfrak{B}$ satisfying certain requirements, and that if $\mathfrak{B}$ is also weakly closed then this occurs if and only if $\mathfrak{A} = \mathfrak{B} \dotplus [\mathfrak{B}^+, \mathfrak{A}]^-$, where $[\mathfrak{B}^+, \mathfrak{A}]^-$ is the norm closure of $\{BX - XB: B \in \mathfrak{B}, B \geq 0 \text{ and } X \in \mathfrak{A}\}$.

Received by the editors February 10, 1977.

AMS (MOS) subject classifications (1970). Primary 46L05, 46L10, 46L25; Secondary 47A20, 47B47.

Key words and phrases. Pure state, singular state, pure state extension, restriction, maximal abelian subalgebra, commutator, compression, conditional expectation, ultrafilter, Stone-Čech compactification, $\beta\mathbf{N}$, rare ultrafilter, p-point, selective ultrafilter.

[1]Partially supported by NSF Grant GP-23392.

In [20] Wils showed that there is a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ so that each singular state on $\mathcal{B}(\mathcal{H})$ may be represented as $\Lambda_{\mathcal{U}}[x_n]$ where $\mathcal{U}$ is an ultrafilter on **N** and for an operator T, $\Lambda_{\mathcal{U}}[x_n](T) = \lim_{\mathcal{U}}(Tx_n, x_n)$. (Recall that a state f on $\mathcal{B}(\mathcal{H})$ is singular if $f(K) = 0$ for each compact operator K on $\mathcal{H}$.) In §4 we give the obvious generalization of this result to arbitrary C^*-algebras. We then use the information developed in the first four sections to study states on $\mathcal{B}(\mathcal{H})$ and states on certain subalgebras of $\mathcal{B}(\mathcal{H})$ in §§5–8. For example in §7 we show that if $\mathcal{C}$ is a maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$, which is isomorphic to $L^\infty(0, 1)$, then there is an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ so that for each nonzero homomorphism h on $\mathcal{C}$ there is an ultrafilter $\mathcal{U}$ on **N** so that $\Lambda_{\mathcal{U}}[e_n]$ agrees with h on $\mathcal{C}$. Thus, if $\mathcal{D}$ is the (maximal abelian) subalgebra of diagonal operators in the basis $\{e_n\}$, there is a pure state f on $\mathcal{B}(\mathcal{H})$ so that f agrees with the homomorphism h on $\mathcal{C}$ and so that f is also a homomorphism on $\mathcal{D}$. We also show that *no* homomorphism of $\mathcal{C}$ has a unique state extension to $\mathcal{B}(\mathcal{H})$.

On the other hand, in §8 we show (assuming the continuum hypothesis) that there are free ultrafilters $\mathcal{U}$ on **N** such that the state $\Lambda_{\mathcal{U}}[e_n]$ is the unique state extension of $\Lambda_{\mathcal{U}}[e_n]|_{\mathcal{D}}$ ($\mathcal{D}$ is the set of diagonal operators in the orthonormal basis $\{e_n\}$) and such that $\Lambda_{\mathcal{U}}[e_n]$ is not a homomorphism on any subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ which is isomorphic to $L^\infty(0, 1)$.

Many of the results in this paper are related to the question: does each nonzero homomorphism of $\mathcal{D}$ have a unique state extension to $\mathcal{B}(\mathcal{H})$? Although we have not been able to settle this question our results do shed some light on the matter. For example, in §8 we show that the method which Reid used to prove that rare ultrafilters have unique state extensions to $\mathcal{B}(\mathcal{H})$ does not work in the general case.

1. The C^*-algebra $\mathcal{M}_f$.

(1.1) $\mathcal{A}$ shall always denote a C^*-algebra containing the identity I. For each state f on $\mathcal{A}$, let $\mathcal{L}_f = \{T \text{ in } \mathcal{A}: f(T^*T) = 0\}$ be the left ideal associated with f. In our discussion, it will be convenient to consider two objects which are closely related to $\mathcal{L}_f$.

DEFINITION. For each state f on $\mathcal{A}$ let

$\mathcal{M}_f = \{T \text{ in } \mathcal{A}: f(TX) = f(XT) = f(T)f(X) \text{ for all } X \text{ in } \mathcal{A}\}$ and

$\mathcal{G}_f = \{T \text{ in } \mathcal{A}: |f(T)| = \|T\| = 1\}$.

Clearly, $\mathcal{M}_f$ is a C^*-algebra containing the identity. In fact, an element T in $\mathcal{A}$ is in $\mathcal{M}_f$ if and only if $T - f(T)I \in \mathcal{L}_f \cap \mathcal{L}_f^*$, where $\mathcal{L}_f^* = \{T \text{ in } \mathcal{A}: T^* \in \mathcal{L}_f\}$. Indeed, if $S = T - f(T)I$, and $S \in \mathcal{L}_f \cap \mathcal{L}_f^*$, then for X in $\mathcal{A}$, $f(SX) = f(XS) = 0 = f(S)f(X)$ by the Cauchy-Schwarz inequality, so $S \in \mathcal{M}_f$ and, hence, $T \in \mathcal{M}_f$. On the other hand, if $S = T - f(T)I \in \mathcal{M}_f$, then $f(S^*S) = f(SS^*) = |f(S)|^2 = 0$, so $S \in \mathcal{L}_f \cap \mathcal{L}_f^*$. Also, by the Cauchy-Schwarz inequality, $\mathcal{G}_f \subset \mathcal{M}_f$, and it follows that $\mathcal{G}_f$ is a semigroup. Note that a projection P (i.e. a selfadjoint idempotent) in $\mathcal{A}$ is in $\mathfrak{M}_f$ if and only if $f(P)$ is 0 or 1, and a unitary U in $\mathcal{A}$ is in $\mathfrak{M}_f$ if and only if $U \in \mathcal{G}_f$. If A is a selfadjoint element of $\mathcal{A}$ such that $(f(A))^2 = f(A^2)$, then $f((A - f(A)I)^2) = 0$, so $A \in \mathfrak{M}_f$. Thus $\mathfrak{M}_f$ is nothing more than the complex span of the "definite" set of f considered by Kadison and Singer **[10]**.

(1.2) If $\mathcal{A}$ is weakly closed and f is a state on $\mathcal{A}$, then $\mathfrak{M}_f$ contains an abundance of projections.

PROPOSITION. *Let f be a state on a von Neumann algebra $\mathcal{A}$ and let A be a selfadjoint element of $\mathfrak{M}_f$ with $f(A) = a$. If δ is a Borel subset of the real line such that a is not in the closure of δ and P_δ is the spectral projection of A associated with δ, then $f(P_\delta) = 0$. In particular, $P_\delta^\perp \in \mathcal{G}_f$.*

PROOF. By the spectral theorem there is an element B in $\mathcal{A}$ so that $B(A - aI) = P_\delta$. Hence, $f(P_\delta) = f(B)f(A - aI) = 0$.

Thus, if f is a state on a von Neumann algebra, $\mathfrak{M}_f$ is the closed linear span of its projections. In some cases one can say even more.

(1.3) EXAMPLE. Let c denote the cardinality of the continuum and let $\{\sigma_\alpha\}_{\alpha\in\kappa}$ be a collection of c infinite subsets of $\mathbf{N}$ such that $\sigma_\alpha \cap \sigma_\beta$ is finite if $\alpha \neq \beta$. For example, identify $\mathbf{N}$ with the rational numbers and let $\{\sigma_\alpha\}$ be subsequences which converge to distinct irrationals. Let $\{e_n\}$ be an orthonormal basis for the separable Hilbert space $\mathcal{H}$ and for each α in κ let P_α be the projection of $\mathcal{H}$ onto $\operatorname{sp}\{e_n: n \in \sigma_\alpha\}$ ($\operatorname{sp}\{\cdot\}$ shall always denote the *closed* linear span of the set within the brackets). If f is a singular state on $\mathcal{B}(\mathcal{H})$, then $f(P_\alpha) = 0$ for all but a countable number of α's. Indeed, if $\{\alpha_1, \ldots, \alpha_n\}$ is a collection of indices, then $P_{\alpha_1} + P_{\alpha_2} + \cdots + P_{\alpha_n} \leq I + K$ where $K = \sum_{i<j} P_{\alpha_i}P_{\alpha_j}$. Since $\sigma_{\alpha_i} \cap \sigma_{\alpha_j}$ is finite if $i \neq j$, $P_{\alpha_i}P_{\alpha_j}$ has finite rank if $i \neq j$ and K is a compact operator. Thus, $f(P_{\alpha_1}) + \cdots + f(P_{\alpha_n}) \leq 1$ and so there are at most n α's for which $f(P_\alpha) \geq 1/n$.

Note that this argument shows that if f is a singular state on $\mathcal{B}(\mathcal{H})$ and P is a projection on $\mathcal{H}$ such that $f(P) = 1$, then there is a projection $Q \leq P$ so that $P - Q$ has infinite rank (P has infinite rank since $f(P) = 1$) and $f(Q) = 1$.

(1.4) Of course, there are no nonzero complex homomorphisms on $\mathcal{B}(\mathcal{H})$ as can be seen in a variety of ways. For example, this follows from the fact that commutators (i.e., operators of the form $[X, Y] = XY - YX$) are norm dense in $\mathcal{B}(\mathcal{H})$ **[4]**. In fact, if f is a nonzero homomorphism on an AW^*-algebra $\mathcal{A}$, then $\mathcal{A}$ has an abelian direct summand. For if $\mathcal{A}$ does not have an abelian direct summand, then the real vector space generated by products of projections in $\mathcal{A}$ is all of $\mathcal{A}$ **[2]**. Hence, if $\mathcal{A}$ is an AW^*-algebra with no abelian direct summand, and f is a homomorphism of $\mathcal{A}$, then f is real on $\mathcal{A}$, so f must be 0.

(1.5) REMARK. Let f be a state on $\mathcal{A}$ and let $\{\pi_f, \mathcal{H}_f, I_f\}$ denote the representation of $\mathcal{A}$ which f induces via the Gelfand-Naimark-Segal construction. Then $\mathcal{M}_f$ induces a natural decomposition of $\mathcal{H}_f$ as follows. Let

$$\mathcal{H}_1 = \{x \in \mathcal{H}_f: \pi_f(T)x = f(T)x \text{ for all } T \in \mathcal{M}_f\}$$

and let

$$\mathcal{H}_2 = \mathrm{sp}\{\pi_f([T, X])I_f: T \in \mathcal{M}_f, X \in \mathcal{A}\}.$$

Then a vector x in $\mathcal{H}_f$ is orthogonal to $\mathcal{H}_2$ if and only if

$$0 = (x, \pi_f([T, X])I_f) = (\pi_f(T^* - f(T^*)I)x, \pi_f(X)I_f)$$

for all T in $\mathcal{M}_f$ and all X in $\mathcal{A}$. This occurs if and only if $\pi_f(T)x = f(T)x$ for all T in $\mathcal{M}_f$ so $\mathcal{H}_1 \perp \mathcal{H}_2$ and $\mathcal{H}_f = \mathcal{H}_1 \oplus \mathcal{H}_2$.

2. The map α_f.

(2.1) DEFINITION. Let f be a state on a C^*-algebra $\mathcal{A}$. For each X in $\mathcal{A}$, let

$\alpha_f(X) = \inf\{\|TXT^*\|: T \in \mathcal{G}_f\}$,

$\beta_f(X) = \inf\{\|AXA\|: A \in \mathcal{G}_f^+\}$,

$\gamma_f(X) = \inf\{\|PXP\|: P \in \mathcal{G}_f^p\}$,

where, $\mathcal{G}_f$ is as in (1.1), $\mathcal{G}_f^+$ denotes the positive elements of $\mathcal{G}_f$, and $\mathcal{G}_f^p$ denotes the projections in $\mathcal{G}_f$.

(2.2) LEMMA. *If f is a state on a C^*-algebra $\mathcal{A}$ then for each X in $\mathcal{A}$, $\alpha_f(X) = \beta_f(X) \leq \gamma_f(X)$. If $\mathcal{A}$ is a von Neumann algebra, then $\alpha_f = \beta_f = \gamma_f$.*

PROOF. By definition $\alpha_f \leq \beta_f \leq \gamma_f$. Let X be in $\mathcal{A}$ and fix $\varepsilon > 0$. Choose T in $\mathcal{G}_f$ so that $\|TXT^*\| < \alpha_f(X) + \varepsilon$. Then, since $\mathcal{G}_f$ is a semigroup,

$$A = T^*T \in \mathcal{G}_f^+ \quad \text{and} \quad \|AXA\| \leq \|T^*\|\,\|TXT^*\|\,\|T\| < \alpha_f(X) + \varepsilon.$$

Thus, since ε was arbitrary, $\beta_f(X) \leq \alpha_f(X)$ and so $\alpha_f = \beta_f$. Now suppose that $\mathcal{A}$ is a von Neumann algebra. Fix $\varepsilon > 0$ and X in $\mathcal{A}$ and choose A in $\mathcal{G}_f^+$ so that $\|AXA\| < \beta_f(X) + \varepsilon/2$. If $0 < r < 1$, let P_r be the spectral projection of A associated with the interval $[1 - r, 1]$. Then since $A \in \mathcal{M}_f$ and $f(A) = 1$, $P_r \in \mathcal{G}_f^p$ by (1.2). Also, by the spectral theorem, $\|P_rA - P_r\| < r$. Hence,

$$\begin{aligned}\|P_rXP_r\| &\leq \|P_rXP_r - P_rAXP_r\| + \|P_rAXP_r - P_rAXAP_r\| + \|P_rAXAP_r\| \\ &\leq 2r\|X\| + \|AXA\| \leq 2r\|X\| + \varepsilon/2 + \beta_f(X).\end{aligned}$$

Thus, since r was arbitrary $\gamma_f(X) \leq \beta_f(X) + \varepsilon$ and, as before, $\gamma_f(X) = \beta_f(X)$.

(2.3) THEOREM. *If f is a state on a C^*-algebra $\mathcal{A}$, then $\alpha_f^{-1}(0) = \{X \text{ in } \mathcal{A}: \alpha_f(X) = 0\} = \mathcal{L}_f + \mathcal{L}_f^*$.*

PROOF. We first show $\mathcal{L}_f \subset \alpha_f^{-1}(0)$. Let T be in $\mathcal{L}_f$ and fix $\varepsilon > 0$. Let B^+ and B^- be the positive and negative parts of $T^*T - \varepsilon I$, respectively. Then B^+ and B^- are in $\mathcal{M}_f$ since T^*T is in $\mathcal{M}_f$ so

$$0 = (f(T^*T))^2 = (f(B^+ + (\varepsilon I - B^-)))^2$$
$$= (f(B^+))^2 + 2f(B^+)f(\varepsilon I - B^-) + (f(\varepsilon I - B^-))^2.$$

Since $B^+ \geqslant 0$ and $\varepsilon I - B^- \geqslant 0$, $f(B)^+ = 0$ and $f(B^-) = \varepsilon = \|B^-\|$. Then $A = \varepsilon^{-1}B^- \in \mathcal{G}_f^+$ and $\|AT^*TA\| < \varepsilon$. Hence,

$$\|ATA\| \leqslant \|TA\| = \|AT^*TA\|^{1/2} \leqslant \varepsilon^{1/2}.$$

Since ε was arbitrary, $\alpha_f(T) = 0$. Let T_1 and T_2 be in $\mathcal{L}_f$. We show that $T_1 + T_2^* \in \alpha_f^{-1}(0)$. Fix $\varepsilon > 0$ and choose A_1 in $\mathcal{G}_f^+$ so that $\|A_1T_1A_1\| < \varepsilon/2$. Then $A_1 \in \mathfrak{M}_f$ so $A_1T_2A_1 \in \mathcal{L}_f$ and there is A_2 in $\mathcal{G}_f^+$ so that $\|A_2A_1T_2A_1A_2\| < \varepsilon/2$. Let $S = A_2A_1$. Then $S \in \mathcal{G}_f$ ($\mathcal{G}_f$ is a semigroup) and

$$\|S(T_1^* + T_2)S^*\| < \|ST_1S^*\| + \|ST_2S^*\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $\alpha_f(T_1 + T_2^*) = 0$. Conversely, suppose $X \in \mathcal{A}$ and $\alpha_f(X) = 0$. Fix $\varepsilon > 0$ and choose A in $\mathcal{G}_f^+$ so that $\|AXA\| < \varepsilon$. Then $f(A)I - A = I - A \in \mathcal{L}_f \cap \mathcal{L}_f^*$. Let $T_1 = AX(I - A)$ and $T_2 = X^*(I - A)$. Then T_1 and T_2 are in $\mathcal{L}_f$ and

$$\|X - (T_1 + T_2^*)\| = \|X - (AX - AXA + X - AX)\| = \|AXA\| < \varepsilon.$$

Hence, X is in the norm closure of $\mathcal{L}_f + \mathcal{L}_f^*$. But by a theorem of Rudin **[18]**, $\mathcal{L}_f + \mathcal{L}_f^*$ is norm closed and the theorem is proved.

(2.4) COROLLARY. *If f is a state on a C^*-algebra $\mathcal{A}$ then f is a pure state if and only if $\alpha_f(X - f(X)I) = 0$ for each X in $\mathcal{A}$.*

PROOF. f is a pure state if and only if $\mathcal{L}_f + \mathcal{L}_f^*$ is the null space of f **[9]**.

(2.5) COROLLARY. *Let f be a state on a C^*-algebra $\mathcal{A}$ and let f_1 be the restriction of f to $\mathfrak{M}_f$. Then f is a pure state if and only if f is the unique state extension of f_1 to $\mathcal{A}$.*

PROOF. Since f_1 is a homomorphism it is a pure state on $\mathfrak{M}_f$. By a familiar argument (see [7, 2.10.1], for example) an extreme point of the set of all state extensions of f_1 to $\mathcal{A}$ must be a pure state. Hence, if f is the unique state extension of f_1 to $\mathcal{A}$, it is a pure state. Conversely, suppose that f is a pure state on $\mathcal{A}$ and that g is a state on $\mathcal{A}$ which agrees with f on $\mathfrak{M}_f$. Then $\mathcal{G}_f \subset \mathcal{G}_g$. Fix X in $\mathcal{A}$ and for each integer n choose A_n in $\mathcal{G}_f$ so that $\|A_n(X - f(X)I)A_n\| < 1/n$ (by (2.4)). Then

$$g(X) = \lim_n g(A_nXA_n) = \lim_n f(X)g(A_n)^2 = f(X).$$

(2.6) We now give several characterizations of singular pure states on $\mathcal{B}(\mathcal{H})$. If f is a state on a C^*-algebra $\mathcal{A}$, let $\mathfrak{N}(f)$ be the null space of f and let $\mathcal{L}_f^+$ be the set of positive elements in $\mathcal{L}_f$. If $\mathcal{S}$ is a subset of $\mathcal{A}$, let $[\mathcal{S}, \mathcal{A}]^-$ denote the norm closure of $\{TX - XT: T \in \mathcal{S}$ and $X \in \mathcal{A}\}$.

THEOREM. *If f is a singular state on $\mathcal{B}(\mathcal{H})$ then the following are equivalent*:
(a) *f is a pure state,*
(b) $[\mathfrak{M}_f, \mathcal{B}(\mathcal{H})]^- = \mathfrak{N}(f)$, *and*
(c) $[\mathcal{L}_f^+, \mathcal{B}(\mathcal{H})]^- = \mathfrak{N}(f)$.

PROOF. Assume (b) is true. Let T be in $\mathfrak{M}_f$. Then $T - f(T)I \in \mathcal{L}_f \cap \mathcal{L}_f^*$ and $[T, X] = [T - f(T)I, X]$ for each X in $\mathcal{B}(\mathcal{H})$. Hence, since $\mathcal{L}_f + \mathcal{L}_f^*$ is norm closed, $\mathfrak{N}(f) = [\mathcal{L}_f \cap \mathcal{L}_f^*, \mathcal{B}(\mathcal{H})]^- \subset \mathcal{L}_f + \mathcal{L}_f^* \subset \mathfrak{N}(f)$, and as in the proof of (2.4), it follows that f is a pure state. Thus, (b) $\Rightarrow$ (a). Since the implication from (c) to (b) is trivial, it only remains to show (a) $\Rightarrow$ (c). Hence, assume f is a pure state on $\mathcal{B}(\mathcal{H})$. Let X be in $\mathfrak{N}(f)$ and fix $\varepsilon > 0$. Then since f is a pure state and $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra there is a projection P in $\mathcal{G}_f$ so that $\|PXP\| < \varepsilon$ ((2.2) and (2.4)). Let $X_1 = X - PXP$. We show that $X_1 \in [\mathcal{L}_f^+, \mathcal{B}(\mathcal{H})]$. Since f is a singular state, as noted in (1.3) there is a projection $Q \leqslant P$ so that $P - Q$ has infinite rank and $f(Q) = 1$. Let X_2 be the restriction of $Q^\perp X_1 Q^\perp$ to $Q^\perp\mathcal{H}$. Then the projection $PQ^\perp$ has infinite rank in $Q^\perp\mathcal{H}$ and $PQ^\perp X_2 PQ^\perp = 0$, so by [**1**, 7.2] there are operators A_1 and Y_1 in $\mathcal{B}(Q^\perp\mathcal{H})$ so that A_1 is positive and invertible and $[A_1, Y_1] = X_2$. Let $A = A_1 \oplus 0$ and $Y = Y_1 \oplus 0 + (A_1^{-1} \oplus 0)X_1Q - QX_1(A_1^{-1} \oplus 0)$, where the 0 direct summands act on $Q\mathcal{H}$. Then

$$[A, Y] = Q^\perp[A_1, Y_1]Q^\perp + Q^\perp X_1 Q + QX_1Q^\perp = X_1 - QX_1Q = X_1.$$

Hence, $\|X - [A, Y]\| = \|PXP\| < \varepsilon$. Also, $f(A) = f(AQ^\perp) = 0$ so $A \in \mathcal{L}_f^+$ and the theorem is proved.

(2.7) If f is a state on a C^*-algebra $\mathcal{A}$, and $X \in \mathcal{A}$, let $G_f(X)$ denote the norm closure of the convex hull of $\{TXT^*: T \in \mathcal{G}_f\}$, and let $G_f^u(X)$ denote the norm closure of the convex hull of $\{UXU^*: U$ is unitary and $U \in \mathcal{G}_f\}$. Note that f is constant on $G_f(X)$.

THEOREM. *If f is a singular state on $\mathcal{B}(\mathcal{H})$ then the following are equivalent*:
(a) *f is a pure state,*
(b) $G_f(X) \cap \mathbf{C}I = \{f(X)I\}$, *for all X in $\mathcal{B}(\mathcal{H})$ and*
(c) $G_f^u(X) \cap \mathbf{C}I = \{f(X)I\}$.

PROOF. Assume that (b) is true and let g be a state on $\mathcal{B}(\mathcal{H})$ which agrees with f on $\mathfrak{M}_f$. Then $\mathcal{G}_g \supset \mathcal{G}_f$ so g is constant on $G_f(X)$ and, by (b), $g(X) = f(X)$. Hence, f is a pure state by (2.5) and (b) $\Rightarrow$ (a). Since the implication from (c) to (b) is trivial, it only remains to show that (a) $\Rightarrow$ (c). Hence, assume that f is a pure state. Since f is constant on $G_f^u(X)$ we need only show $f(X)I \in G_f^u(X)$. Let X be in $\mathcal{B}(\mathcal{H})$ and fix $\varepsilon > 0$. By (2.6) there are operators A and Y so that $A \in \mathcal{L}_f^+$ and $\|X - f(X)I - [A, Y]\| < \varepsilon/2$. By the spectral theorem and (1.2) there are mutually orthogonal projections

$P_1, P_2, \ldots, P_n$ and real numbers $0 < a_1 < a_2 < \cdots < a_n$ so that $P_1 + P_2 + \cdots + P_n = I$, $f(P_1) = 1$, $P_k A = AP_k$ for $k = 1, 2, \ldots, n$, and $\|P_k A - a_k P_k\| < \varepsilon(4\|Y\|)^{-1}$ for $k = 1, 2, \ldots, n$. Let $B = P_1 + 2P_2 + \cdots + nP_n$ and let $Z = \sum_{j \neq k} (j-k)^{-1} P_j[A, Y]P_k$. Then $[B, Z] = \sum_{j \neq k} P_j[A, Y]P_k$ and

$$\|[A, Y] - [B, Z]\| = \max_k \|P_k[A, Y]P_k\|$$
$$= \max_k \|[AP_k - a_k P_k, P_k Y P_k]\| < \frac{\varepsilon}{2}.$$

Let $U_1 = P_1 - (I - P_1)$. Then U_1 is unitary, $U_1 \in \mathcal{G}_f(f(P_1) = 1)$ and

$$\tfrac{1}{2} U_1[B, Z]U_1 + \tfrac{1}{2}[B, Z] = (I - P_1)[B, Z](I - P_1).$$

Let $U_2 = P_1 + P_2 - (I - P_1 - P_2)$. Then U_2 is unitary, $U_2 \in \mathcal{G}_f$ and

$$\tfrac{1}{2} U_2(I - P_1)[B, Z](I - P_1)U_2 + \tfrac{1}{2}(I - P_1)[B, Z](I - P_1)$$
$$= (I - P_1 - P_2)[B, Z](I - P_1 - P_2).$$

Applying this procedure $n - 2$ more times, we obtain unitaries $V_1, V_2, \ldots, V_m$ and real numbers $t_1, t_2, \ldots, t_m$ so that $0 \leqslant t_j \leqslant 1$ for $1 \leqslant j \leqslant m$, $t_1 + t_2 + \cdots + t_m = 1$, $V_j \in \mathcal{G}_f$ for $j = 1, 2, \ldots, m$ and $\sum_{j=1}^m t_j V_j[B, Z]V_j^* = 0$. Hence,

$$\|\Sigma\, t_j V_j X V_j^* - f(X)I\| = \|\Sigma\, t_j V_j(X - f(X)I - [B, Z])V_j^*\|$$
$$\leqslant \|X - f(X)I - [B, Z]\|$$
$$\leqslant \|X - f(X)I - [A, Y]\| + \|[A, Y] - [B, Z]\|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since ε was arbitrary, $f(X)I \in G_f^u(X)$, (a) $\Rightarrow$ (c), and the theorem is proved.

(2.8) REMARK. If f is a singular state on $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}_f$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are as in (1.5), then f is a pure state if and only if $\pi_f(\mathfrak{M}_f)|_{\mathcal{H}_2}$ is irreducible. Indeed, if f is a pure state, the transitivity theorem [7, 2.8.3] shows that $\pi_f(\mathfrak{M}_f)|_{\mathcal{H}_2}$ is irreducible. Conversely, if $\pi_f(\mathfrak{M}_f)|_{\mathcal{H}_2}$ is irreducible, choose a projection P so that P and $P^\perp$ have infinite rank and $f(P) = 1$. Let W be a partial isometry so that $W^*W = P$ and $WW^* = P^\perp$, and let $g = f(W^* \cdot W)$. Then $g(P^\perp) = f(P) = 1$ so g is a state. Let h be the restriction of g to $\mathfrak{M}_f$. The map $\pi_h(T)I_h \mapsto \pi_f(T)\pi_f(W)I_f$ determines a unitary transformation U such that $U\pi_h(\mathfrak{M}_f)U^* = \pi_f(\mathfrak{M}_f)|_{\mathcal{H}_2}$. Hence, $\pi_h(\mathfrak{M}_f)$ is irreducible and h is a pure state on $\mathfrak{M}_f$. If X is an operator, then $P^\perp X P^\perp \in \mathfrak{M}_f$, so $g(X) = h(P^\perp X P^\perp)$. Thus, g is the unique extension of h to $\mathcal{B}(\mathcal{H})$. Hence, by (2.5) g is a pure state on $\mathcal{B}(\mathcal{H})$ and so $f = g(W \cdot W^*)$ is a pure state.

3. Uniqueness of state extensions.

(3.1) Throughout this section $\mathcal{A}$ and $\mathcal{B}$ shall always denote C^*-algebras such that $I \in \mathcal{B} \subset \mathcal{A}$. We first characterize the pure states f on $\mathcal{B}$ which have unique state extensions to $\mathcal{A}$ in terms of compressions of elements of $\mathcal{A}$ by elements of $\mathcal{G}_f$.

DEFINITION. Let f be a state on $\mathcal{B}$. We say that $\mathcal{A}$ is $\mathcal{B}$*-compressible modulo* f if for each X in $\mathcal{A}$ and each $\varepsilon > 0$ there is B in $\mathcal{G}_f^+$ and Y in $\mathcal{B}$ so that $\|BXB - Y\| < \varepsilon$.

Note that if f is any state on $\mathcal{B}$ such that $\mathcal{A}$ is $\mathcal{B}$-compressible modulo f then f has a unique state extension to $\mathcal{A}$. For if g and h are states on $\mathcal{A}$ which agree with f on $\mathcal{B}$ and $X \in \mathcal{A}$, we may choose B_n in $\mathcal{G}_f^+$ and Y_n in $\mathcal{B}$ so that $\|B_nXB_n - Y_n\| \to 0$. Since g and h agree with f on $\mathcal{B}$, $\mathcal{G}_f \subset \mathcal{G}_g \cap \mathcal{G}_h$ and

$$g(X) = \lim_n g(B_nXB_n) = \lim_n f(Y_n) = \lim_n h(B_nXB_n) = h(X).$$

(3.2) THEOREM. *If f is a pure state on $\mathcal{B}$ then f has a unique (pure) state extension to $\mathcal{A}$ if and only if $\mathcal{A}$ is $\mathcal{B}$-compressible modulo f. In fact, if g is the unique pure state extension of f to $\mathcal{A}$ and $X \in \mathcal{A}$, then for each $\varepsilon > 0$ there is B in $\mathcal{G}_f^+$ so that $\|B(X - g(X)I)B\| < \varepsilon$. If $\mathcal{B}$ is a von Neumann algebra, B may be taken to be a projection.*

PROOF. As noted above if $\mathcal{A}$ is $\mathcal{B}$-compressible modulo f, then f has a unique state extension to $\mathcal{A}$ which must be a pure state [7, 2.10.1]. Conversely, suppose that g is the unique pure state extension of f to $\mathcal{A}$. Then, by [7, 2.10.1] for each selfadjoint element A in $\mathcal{A}$, $g(A) = \sup\{f(B): B \in \mathcal{B}, B \leqslant A\}$. Let X be in $\mathcal{A}$ and fix $\varepsilon > 0$. We will show that there is B in $\mathcal{G}_f^+$ so that $\|B(X - g(X)I)B\| < \varepsilon$. Since g is a pure state, $X = T_1 + T_2^* + g(X)I$, where T_1 and T_2 are in $\mathcal{L}_g$. Let $A_1 = T_1^*T_1$ and choose B_1 in $\mathcal{B}$ so that $-B_1 \leqslant -A_1$ and $f(-B_1) + r > g(-A_1) = 0$ where $0 < 2(2r)^{1/2} < \varepsilon$. Then, $0 \leqslant A_1 \leqslant B_1$ and $r > f(B_1) \geqslant 0$. Since f is a pure state on $\mathcal{B}$, $\alpha_f(B_1 - f(B_1)I) = 0$ by (2.4). Hence, there is C_1 in $\mathcal{G}_f^+$ so that $\|C_1(B_1 - f(B_1)I)C_1\| < r$. Then $0 \leqslant C_1A_1C_1 \leqslant C_1B_1C_1$ so

$$\|C_1A_1C_1\| \leqslant \|C_1B_1C_1\| \leqslant \|C_1(B_1 - f(B_1)I)C_1\| + f(B_1) < 2r.$$

Since $C_1 \in \mathcal{M}_f \subset \mathcal{M}_g$, $T_2C_1 \in \mathcal{L}_g$. Let $A_2 = C_1T_2^*T_2C_1$. Then, as above, there is C_2 in $\mathcal{G}_f^+$ so that $\|C_2A_2C_2\| < 2r$. Let $B = C_1C_2^2C_1$. Then $B \in \mathcal{G}_f^+$ and

$$\|B(X - g(X)I)B\| = \|B(T_1 + T_2^*)B\| \leqslant \|BT_1B\| + \|BT_2^*B\|$$
$$\leqslant \|T_1B\| + \|T_2B\| = \|BT_1^*T_1B\|^{1/2} + \|BT_2^*T_2B\|^{1/2} \leqslant 2(2r)^{1/2}.$$

If $\mathcal{B}$ is a von Neumann algebra, as in (2.2) we may choose a spectral projection P of B so that $P \in \mathcal{G}_f$ and $\|PB - P\|$ is small. It then follows (as in (2.2)) that $\|P(X - g(X)I)P\|$ is small and the theorem is proved.

(3.3) We say that $\mathcal{B}$ has the *extension property relative to* $\mathcal{A}$ if each pure state of $\mathcal{B}$ has a unique pure state extension to $\mathcal{A}$. We now restrict ourselves to the case where $\mathcal{B}$ is maximal abelian in $\mathcal{A}$ and give several characterizations of C^*-algebras $\mathcal{B}$ with the extension property relative to $\mathcal{A}$.

Recall that a conditional expectation Θ is a linear map of $\mathcal{A}$ onto $\mathcal{B}$ satisfying (i) $\Theta(A_1) \leqslant \Theta(A_2)$ if $A_1, A_2 \in \mathcal{A}$ and $A_1 \leqslant A_2$, (ii) $\Theta(TX) = \Theta(T)\Theta(X)$ and $\Theta(XT) = \Theta(X)\Theta(T)$ for all T in $\mathcal{B}$ and all X in $\mathcal{A}$, (iii) $\Theta(T) = T$ for all T in $\mathcal{B}$ and, (iv) $\|\Theta\| = 1$.

(3.4) THEOREM. *If $\mathcal{B}$ is a maximal abelian subalgebra of $\mathcal{A}$ then $\mathcal{B}$ has the extension property relative to $\mathcal{A}$ if and only if there is a conditional expectation Θ of $\mathcal{A}$ onto $\mathcal{B}$ such that for each homomorphism h of $\mathcal{B}$ and each state f on $\mathcal{A}$ which agrees with h on $\mathcal{B}$, $f = h \circ \Theta$.*

PROOF. Clearly, if Θ is such a conditional expectation, $\mathcal{B}$ has the extension property relative to $\mathcal{A}$. Conversely, suppose each homomorphism h of $\mathcal{B}$ has a unique pure state extension f_h on $\mathcal{A}$. Let $\mathcal{P}(\mathcal{B})$ be the set of homomorphisms of $\mathcal{B}$. Since $\mathcal{B}$ is abelian, $\mathcal{P}(\mathcal{B})$ with the weak*-topology of $\mathcal{B}$ is a compact Hausdorff space and the Gelfand transform Ψ is an isometric isomorphism of $\mathcal{B}$ onto $C(\mathcal{P}(\mathcal{B}))$, the continuous functions on $\mathcal{P}(\mathcal{B})$. We define a map Φ of $\mathcal{A}$ onto $C(\mathcal{P}(\mathcal{B}))$ by $\Phi(X)(h) = f_h(X)$ for each X in $\mathcal{A}$ and each h in $\mathcal{P}(\mathcal{B})$. Since f_h is the unique extension of h, $\Phi(X)$ is well defined for each X in $\mathcal{A}$. We now show that $\Phi(X)$ is continuous. Fix h in $\mathcal{P}(\mathcal{B})$ and X in $\mathcal{A}$. Let ε and r be positive real numbers and let $\Phi(X)(h) = \lambda$. Since f_h is the unique extension of the pure state h, by (3.2) there is B in $\mathcal{G}_h^+$ so that $\|B(X - f_h(X)I)B\| < r$. Let $V = \{k \text{ in } \mathcal{P}(\mathcal{B})\colon k(B) = \Psi(B)(k) > 1 - r\}$. Then if $k \in V$,

$$\begin{aligned} |\Phi(X)(k) - \lambda| &= |f_k(X) - \lambda| \leqslant |f_k(X) - f_k(BXB)| + |f_k(BXB) - \lambda| \\ &\leqslant |1 - (f_k(B))^2|\,|f_k(X)| + |f_k(BXB) - \lambda(f_k(B))^2| \\ &\quad + |\lambda|\,|1 - (f_k(B))^2| \\ &\leqslant 2|1 - f_k(B)|(|f_k(X)| + |\lambda|) + |f_k(B(X - \lambda I)B)| \\ &\leqslant 2r(\|X\| + |\lambda|) + r. \end{aligned}$$

Hence if r is chosen appropriately, $\Phi(X)$ maps the open set V (which contains h) into $\{\mu \in \mathbf{C}\colon |\mu - \lambda| < \varepsilon\}$, so $\Phi(X)$ is continuous. Hence, $\Phi(X) = \Psi(T)$ for some unique T in $\mathcal{B}$. Define Θ on $\mathcal{A}$ by $\Theta = \Psi^{-1}\Phi$. It is easy to see that Θ is a conditional expectation. For example, if $T \in \mathcal{B}$ and $X \in \mathcal{A}$, then for each h in $\mathcal{P}(\mathcal{B})$,

$$\begin{aligned} h(\Theta(TX)) &= \Psi(\Theta(TX))(h) = \Phi(TX)(h) = f_h(TX) \\ &= f_h(T)f_h(X) = h(\Theta(T)\Theta(X)). \end{aligned}$$

Since $\mathcal{P}(\mathcal{B})$ separates points in $\mathcal{B}$, $\Theta(TX) = \Theta(T)\Theta(X)$. Note that $f_h(X) = \Phi(X)(h) = \Psi\Theta(X)(h) = h(\Theta(X))$; i.e., $f_h = h \circ \Theta$. The theorem is proved.

Note that if $\mathcal{B}$ is not maximal abelian in $\mathcal{A}$ then $\mathcal{B}$ is contained in a maximal abelian subalgebra $\mathcal{B}_1$ of $\mathcal{A}$ and by the Stone-Weierstrass theorem $\mathcal{B}$ does not even have the extension property relative to $\mathcal{B}_1$.

(3.5) Note the connection between (3.4) and Lemmas 1 and 2 in **[10]**. We now give a generalization of Lemma 5 in **[10]**.

DEFINITION. Let $\mathfrak{B}$ be a maximal abelian subalgebra of $\mathfrak{A}$ and let h be a nonzero homomorphism on $\mathfrak{B}$. We say that a subset $\{B_1, \ldots, B_n\}$ is a *partition of unity subordinate to h* if $0 \leqslant B_i \leqslant I$ for $i = 1, 2, \ldots, n$, $B_1 + B_2 + \cdots + B_n = I$ and $h(B_1) = 1$, $h(B_2) = \cdots = h(B_n) = 0$. We say that $\mathfrak{A}$ is $\mathfrak{B}$*-compressible* if for each nonzero homomorphism h on $\mathfrak{B}$, for each X in $\mathfrak{A}$ and each $\varepsilon > 0$, there is a partition of unity $\{B_1^2, \ldots, B_n^2\}$ subordinate to h and Y in $\mathfrak{B}$ so that $\|B_i(X - Y)B_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$. We say that $\mathfrak{A}$ is *orthogonally* $\mathfrak{B}$*-compressible* if $\mathfrak{A}$ is $\mathfrak{B}$-compressible and the partition of unity can always be taken to consist of mutually orthogonal projections.

(3.6) THEOREM. *If $\mathfrak{B}$ is a maximal abelian subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ has the extension property relative to $\mathfrak{A}$ if and only if $\mathfrak{A}$ is $\mathfrak{B}$-compressible. If $\mathfrak{B}$ is a weakly closed maximal abelian subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ has the extension property relative to $\mathfrak{A}$ if and only if $\mathfrak{A}$ is orthogonally $\mathfrak{B}$-compressible.*

PROOF. The proof is essentially the same as the proof of Lemma 5 in **[10]**. Clearly, if $\mathfrak{A}$ is $\mathfrak{B}$-compressible, then $\mathfrak{A}$ is $\mathfrak{B}$-compressible modulo h for each h in $\mathfrak{P}(\mathfrak{B})$, the nonzero homomorphisms on $\mathfrak{B}$, and so by (3.2) $\mathfrak{B}$ has the extension property relative to $\mathfrak{A}$. Suppose that $\mathfrak{B}$ has the extension property relative to $\mathfrak{A}$. Let X be in $\mathfrak{A}$, let h_0 be in $\mathfrak{P}(\mathfrak{B})$ and let ε be a positive real number. We shall construct a partition of unity with the required properties. By (3.4) there is a conditional expectation Θ of $\mathfrak{A}$ onto $\mathfrak{B}$. Let $X_1 = X - \Theta(X)$ and choose a real number r so that $0 < 4r\|X_1\| + r < \varepsilon$. By (3.2) for each homomorphism h of $\mathfrak{B}$, there is C_h' in $\mathcal{G}_h^+$ so that $\|C_h'X_1C_h'\| < r$ $(f_h(X_1) = h(\Theta(X_1)) = 0)$. Let $V_h = \{k \text{ in } \mathfrak{P}(\mathfrak{B})\colon k(C_h') > 1 - r\}$ and let $W_h = \{k \in \mathfrak{P}(\mathfrak{B})\colon k(C_h') > 1 - 2r\}$. By Urysohn's Lemma there is a positive element C_h in $\mathfrak{B}$ so that $k(C_h) = 1$ if $k \in V_h$, $k(C_h) = 0$ if $k \notin W_h$ and $\|C_h\| = 1$. Then $\|C_h(I - C_h')\| < 2r$ and it follows (as in (2.2)) that $\|C_hX_1C_h\| < 4r\|X_1\| + r < \varepsilon$. Since the collection $\{V_h\}$ is an open cover for $\mathfrak{P}(\mathfrak{B})$, there is a finite subcover $\{V_{h_1}, V_{h_2}, \ldots, V_{h_n}\} \subset \{V_h\}$. Let $V_i = V_{h_i}$ and $C_i = C_{h_i}$ for $i = 1, 2, \ldots, n$. By a standard argument (see **[17**, p. 41**]**, for example) there is a partition of unity $\{B_1^2, \ldots, B_n^2\} \subset \mathfrak{B}$ so that $B_i^2C_i = B_i^2$ for $i = 1, 2, \ldots, n$ and $h_0(B_1^2) = 1$, $h_0(B_2^2) = \cdots = h_0(B_n^2) = 0$. Then for each i, $B_iC_i = B_i$ so $\|B_iX_1B_i\| \leqslant \|C_iX_1C_i\| < \varepsilon$. Thus, $\mathfrak{A}$ is $\mathfrak{B}$-compressible. If $\mathfrak{B}$ is weakly closed and $\mathfrak{A}$ is orthogonally $\mathfrak{B}$-compressible then $\mathfrak{A}$ is $\mathfrak{B}$-compressible so $\mathfrak{B}$ has the extension property relative to $\mathfrak{A}$ by the first part of the proof. Conversely, if $\mathfrak{B}$ is weakly closed and has the extension property relative to $\mathfrak{A}$, then $\mathfrak{P}(\mathfrak{B})$ is extremely disconnected. Hence, arguing as before, we obtain open sets $V_1, \ldots, V_n$ which cover $\mathfrak{P}(\mathfrak{B})$ and elements $C_1, C_2, \ldots, C_n$ of $\mathfrak{B}$ such that $0 \leqslant C_i \leqslant I$, $h(C_i) = 1$ for all h in V_i, and

$\|C_iX_1C_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$. (As before $X_1 = X - \Theta(X)$ for some X in $\mathfrak{A}$.) Then the closure V_i^- of each V_i is open and Q_i, the characteristic function of V_i^-, is in $C(\mathfrak{P}(\mathfrak{B}))$. Furthermore $Q_i \leqslant C_i$, for $1 \leqslant i \leqslant n$. Hence $\|Q_iX_1Q_i\| < \varepsilon$ for each i. Let $P_1 = Q_1$, and $P_i = (P_1 + \cdots + P_{i-1})^\perp Q_i$ for $i = 2, \ldots, n$. Then $\{P_1, \ldots, P_n\}$ is a partition of unity consisting of mutually orthogonal projections and $h_0(P_i) = 1$ for exactly one projection P_i. Since $P_i \leqslant Q_i$ for each i, $\mathfrak{A}$ is orthogonally $\mathfrak{B}$-compressible.

(3.7) For each X in $\mathfrak{A}$ let $G_{\mathfrak{B}}(X)$ be the norm closure of the convex hull of $\{UXU^*$: U is a unitary element in $\mathfrak{B}\}$.

Corollary. *If $\mathfrak{B}$ is a weakly closed maximal abelian subalgebra of $\mathfrak{A}$ then the following are equivalent*:

(a) $\mathfrak{B}$ *has the extension property relative to* $\mathfrak{A}$,

(b) $\mathfrak{B} \dotplus [\mathfrak{B}^+, \mathfrak{A}]^- = \mathfrak{A}$ ($\dotplus$ *denotes the algebra direct sum*) *and*,

(c) $G_{\mathfrak{B}}(X) \cap \mathfrak{B}$ *contains exactly one point for each X in $\mathfrak{A}$*. (*In fact, this point is $\Theta(X)$ where Θ is as in* (3.4).)

Proof. Assume that $\mathfrak{B}$ has the extension property relative to $\mathfrak{A}$. Then by (3.6) $\mathfrak{A}$ is orthogonally $\mathfrak{B}$-compressible, and by (3.4) there is a conditional expectation Θ of $\mathfrak{A}$ onto $\mathfrak{B}$. Let X be in $\mathfrak{A}$ and fix $\varepsilon > 0$. Let $X_1 = X - \Theta(X)$. Then there are mutually orthogonal projections $P_1, P_2, \ldots, P_n$ in $\mathfrak{B}$ so that $P_1 + P_2 + \cdots + P_n = I$ and $\|P_iX_1P_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$. Let $A = P_1 + 2P_2 + \cdots + nP_n$ and let $Y = \Sigma_{i \neq j} (i-j)^{-1}P_iX_1P_j$. Then $A \geqslant 0$, $A \in \mathfrak{B}$ and $[A, Y] = X_1 - \Sigma P_iX_1P_i$. So $\|X - \Theta(X) - [A, Y]\| = \|\Sigma P_iX_1P_i\| = \max\|P_iX_1P_i\| < \varepsilon$. Since ε was arbitrary (a) $\Rightarrow$ (b). Now assume that (b) is true. We show (b) $\Rightarrow$ (c). First note that if f is a state on $\mathfrak{A}$ such that f is a homomorphism on $\mathfrak{B}$ then $\mathfrak{B} \subset \mathfrak{M}_f$ and f is constant on $G_{\mathfrak{B}}(X)$ for each X in $\mathfrak{A}$. Hence, since $\mathfrak{P}(\mathfrak{B})$ separates points of $\mathfrak{B}$, $G_{\mathfrak{B}}(X)$ contains at most one element of $\mathfrak{B}$. So it suffices to show that $G_{\mathfrak{B}}(X) \cap \mathfrak{B} \neq \varnothing$ for each X in $\mathfrak{A}$. This is accomplished by an obvious modification of the proof that (a) $\Rightarrow$ (c) in (2.7). Note that if Θ is a conditional expectation on $\mathfrak{A}$ as in (3.4), and $G_{\mathfrak{B}}(X) \cap \mathfrak{B} = \{T\}$, then for any h in $\mathfrak{P}(\mathfrak{B})$, $h(T) = f_h(X) = h(\Theta(X))$, and since $\mathfrak{P}(\mathfrak{B})$ separates points of $\mathfrak{B}$, $T = \Theta(X)$. Finally, assume that (c) is true. If f and g are states on $\mathfrak{A}$ which agree with a homomorphism h on $\mathfrak{B}$, then $\mathcal{G}_f \cap \mathcal{G}_g \supset \{$unitary elements in $\mathfrak{B}\}$ and so f and g are constant on $G_{\mathfrak{B}}(X)$. Since $G_{\mathfrak{B}}(X) \cap \mathfrak{B} \neq \varnothing, f = g$ and the corollary is proved.

(3.8) Remark. If we do not require that $\mathfrak{B}$ is abelian, then the relationship between the extension property and compressibility is not as clear. In fact, if f is a pure state on a von Neumann algebra $\mathfrak{A}$ such that $\mathfrak{M}_f \neq \mathfrak{A}$, then by (3.2), for each X in $\mathfrak{A}$ and each $\varepsilon > 0$, there is a projection P in $\mathcal{G}_f$ so that $\|P(X - f(X)I)P\| < \varepsilon$. If we let $Y = P^\perp XP^\perp + f(X)P$, then $Y \in \mathfrak{M}_f$, $\|P(X - Y)P\| < \varepsilon$ and $\|P^\perp(X - Y)P^\perp\| = 0$ so that $\mathfrak{A}$ is "$\mathfrak{M}_f$-compressible". However, $\mathcal{M}_f$ does not have the extension property relative to $\mathcal{A}$. Indeed, in the notation of (1.5), $\mathcal{H}_1 = \text{sp}\{I_f\}$ since f is a pure state (by the transitivity theorem) and $\mathcal{H}_2 \neq \{0\}$ since f is not a homomorphism ($\mathcal{M}_f \neq \mathcal{A}$). Let x be a unit vector in H_2 and let $y = (1/2)^{1/2}(I_f + x)$, $z = (1/2)^{1/2}(I_f - x)$. Then the pure states $\omega_y \circ \pi_f$ and $\omega_z \circ \pi_f$ agree on $\mathcal{M}_f$, since $\mathcal{H}_2$ is reducing for $\pi_f(\mathcal{M}_f)$ but are distinct by the transitivity theorem.

4. Wils' Theorem.

(4.1) In his thesis **[20]** Wils showed that there is a fixed sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ so that each singular state on $\mathcal{B}(\mathcal{H})$ has the form $\Lambda_{\mathcal{U}}[x_n]$ for some ultrafilter $\mathcal{U}$ on **N**. In this section we present the obvious generalization of this result to arbitrary C^*-algebras.

Let κ be a set. If we endow κ with the discrete topology, then $\beta\kappa$, the Stone-Čech compactification of κ map be regarded as the set of ultrafilters on κ, where κ is identified with the principal ultrafilters. In particular, for each subset σ of κ the set $W(\sigma) = \{\mathcal{U} \text{ in } \beta\kappa\colon \sigma \in \mathcal{U}\}$ is a closed-open set in $\beta\kappa$ and the family $\{W(\sigma)\}$, where σ ranges over all possible subsets of κ, is a base for this topology. Recall that if φ is a function on κ taking on values in a compact Hausdorff space Δ, then there is a unique continuous map φ_1 of $\beta\kappa$ into Δ which extends φ given by $\varphi_1(\mathcal{U}) = \lim_{\mathcal{U}} \varphi(\alpha) = \bigcap \{\varphi(\sigma)^-\colon \sigma \in \mathcal{U}\}$ for each $\mathcal{U}$ in β.

DEFINITION. Let $\{f_\alpha\}$ be a family of states on a C^*-algebra $\mathcal{A}$ indexed over a set κ. Then $\mathcal{S}(\mathcal{A})$, the set of states on $\mathcal{A}$ provided with the weak*-topology of $\mathcal{A}$ is a compact Hausdorff space and so the map $\alpha \mapsto f_\alpha$ has a unique *continuous* extension to all of $\beta\kappa$. We denote the extended map by $\Lambda[f_\alpha]$ and its value at an ultrafilter $\mathcal{U}$ by $\Lambda_{\mathcal{U}}[f_\alpha]$. Then $\Lambda_{\mathcal{U}}[f_\alpha] = \lim_{\mathcal{U}} f_\alpha$ and it is easy to verify that for each X in $\mathcal{A}$,

$$\Lambda_{\mathcal{U}}[f_\alpha](X) = \lim_{\mathcal{U}} f_\alpha(X) = \bigcap \{F(\sigma)\colon \sigma \in \mathcal{U}\},$$

where $F(\sigma) = \{f_\alpha(X)\colon \alpha \in \sigma\}^-$.

(4.2) Since the map $\Lambda[f_\alpha]$ defined above is continuous, its range is a compact subset of $\mathcal{S}(\mathcal{A})$. It follows that the extreme points of the weak*-closure of the convex hull of the range of $\Lambda[f_\alpha]$ are in the range of $\Lambda[f_\alpha]$. In particular, if convex combinations of elements in the range of $\Lambda[f_\alpha]$ are weak*-dense in $\mathcal{S}(\mathcal{A})$, then the range of Λ contains the pure states of $\mathcal{A}$. As Dixmier has observed [7, 3.4.1], this will occur if the range of $\Lambda[f_\alpha]$ is *total* in the following sense.

DEFINITION. We say that a subset $\mathcal{R}$ of the set of states on a C^*-algebra $\mathcal{A}$ is *total for* $\mathcal{A}$ if for a selfadjoint element A in $\mathcal{A}$, $f(A) \geqslant 0$ for all f in $\mathcal{R}$ implies that $A \geqslant 0$.

(Equivalently, if A is a selfadjoint element of $\mathcal{A}$ which is not positive, then there exists a state f in $\mathcal{R}$ so that $f(A) < 0$.)

Let $\mathcal{S}(\mathfrak{A})$ denote the weak*-closure of the pure states on the C^*-algebra $\mathfrak{A}$. $\mathcal{S}(\mathfrak{A})$ is called the *pure state space* of $\mathfrak{A}$.

(4.3) PROPOSITION. *If $\{f_\alpha\}$ is a family of states on a C^*-algebra $\mathfrak{A}$ indexed over a set κ such that the range $\mathfrak{R}$ of the map $\Lambda[f_\alpha]$ from $\beta\kappa$ into $\mathfrak{S}(\mathfrak{A})$ is total for $\mathfrak{A}$, then $\mathcal{S}(\mathfrak{A}) \subset \mathfrak{R}$. In particular, if $\{f_\alpha\}$ is total for $\mathfrak{A}$, then $\mathcal{S}(\mathfrak{A}) \subset \mathfrak{R}$.*

PROOF. As observed above, $\mathfrak{R}$ is compact. Since $\mathfrak{R}$ is total for $\mathfrak{A}$, by [7, 3.4.1] $\mathfrak{R}^- = \mathfrak{R}$ contains the pure states of $\mathfrak{A}$. Hence, $\mathcal{S}(\mathfrak{A}) \subset \mathfrak{R}$.

If $\{f_\alpha\}$ is total for $\mathfrak{A}$, then $\mathfrak{R}$ is total for $\mathfrak{A}$, because for each β in κ, f_β has the form $\Lambda_{\mathfrak{U}}[f_\alpha]$, where $\mathfrak{U}$ is the principal ultrafilter which contains $\{\beta\}$.

(4.4) We shall use the following characterization of the pure state space of a von Neumann algebra which is due to Glimm [8, Theorem 4′]: if $\mathfrak{A}$ is a von Neumann algebra with center $\mathfrak{Z}$, a state f on $\mathfrak{A}$ is in $\mathcal{S}(\mathfrak{A})$ if and only if f is a homomorphism on $\mathfrak{Z}$ and $f = tf_1 + (1 - t)f_2$, where $0 \leq t \leq 1$, $f_1(E) = 1$ for some abelian projection E in $\mathfrak{A}$ and $f_2(E) = 0$ for all abelian projections E in $\mathfrak{A}$.

If x is a unit vector in a Hilbert space $\mathfrak{H}$, the *vector state* ω_x determined by x is the map $T \mapsto (Tx, x)$ for T in $\mathfrak{B}(\mathfrak{H})$. If $\{x_\alpha\}$ is a family of unit vectors indexed over a set κ we shall denote the map of $\beta\kappa$ into $\mathfrak{S}(\mathfrak{B}(\mathfrak{H}))$ which the collection of vector states determined by the family $\{x_\alpha\}$ induces (as in (4.2)) by $\Lambda[x_\alpha]$. Thus, if $\mathfrak{U} \in \beta\kappa$, $\Lambda_{\mathfrak{U}}[x_\alpha](T) = \lim_{\mathfrak{U}}(Tx_\alpha, x_\alpha)$ for T in $\mathfrak{B}(\mathfrak{H})$.

5. States on $\mathfrak{B}(\mathfrak{H})$.

(5.1) We first assume that $\mathfrak{H}$ is a complex Hilbert space of infinite (but arbitrarily) dimension.

Let $\{x_\alpha\}$ be a set of unit vectors in $\mathfrak{H}$ indexed over a set κ. The map $\alpha \mapsto x_\alpha$ for each α in κ extends uniquely to a continuous function of $\beta\kappa$ into the unit ball of $\mathfrak{H}$ with its weak topology. Hence for each ultrafilter $\mathfrak{U}$ in $\beta\kappa$ there is a unique vector y in $\mathfrak{H}$ such that $\|y\| \leq 1$ and $\lim_{\mathfrak{U}}(x_\alpha, z) = (y, z)$ for all vectors z in $\mathfrak{H}$. We denote this relationship by $x_\alpha \rightharpoonup y(\mathfrak{U})$. It is readily verified that if $x_\alpha \rightharpoonup y(\mathfrak{U})$, then $\|y\| = 1$ if and only if $\lim_{\mathfrak{U}}\|x_\alpha - y\| = 0$. We denote this circumstance by $x_\alpha \to y(\mathfrak{U})$.

A projection in $\mathfrak{B}(\mathfrak{H})$ is abelian if and only if it has rank 1. Thus, a state on $\mathfrak{B}(\mathfrak{H})$ takes the value 1 at an abelian projection if and only if it is a vector state, and a state annihilates the abelian projections if and only if it annihilates the compact operators on $\mathfrak{H}$, i.e. if and only if it is a singular state. Since the center of $\mathfrak{B}(\mathfrak{H})$ is trivial, by Glimm's theorem $\mathcal{S}(\mathfrak{B}(\mathfrak{H}))$ consists of convex combinations of vector states and singular states. We now use these facts to show that $\mathcal{S}(\mathfrak{B}(\mathfrak{H}))$ is the continuous image of $\beta\mathbf{N}$.

We shall say that a set of unit vectors $\{x_\alpha\}$ is total for $\mathfrak{B}(\mathfrak{H})$ if the associated vector states are total for $\mathfrak{B}(\mathfrak{H})$. Elements of $\mathfrak{B}(\mathfrak{H})$ shall be called operators.

(5.2) LEMMA. *If $\{x_\alpha\}$ is a set of unit vectors in $\mathcal{H}$ indexed over a set κ and $\mathcal{U}$ is an ultrafilter on κ, then the state $\Lambda_{\mathcal{U}}[x_\alpha]$ is a singular state if and only if $x_\alpha \rightharpoonup O(\mathcal{U})$.*

PROOF. Let y and z be vectors in $\mathcal{H}$ and let $y \otimes z$ be the operator which maps the vector x to $(x, z)y$. Then finite linear combinations of operators of this form are uniformly dense in the set of compact operators on $\mathcal{H}$. Hence, $\Lambda_{\mathcal{U}}[x_\alpha]$ is a singular state if and only if $\Lambda_{\mathcal{U}}[x_\alpha](y \otimes z) = 0$ for all vectors y and z in $\mathcal{H}$ and this occurs if and only if $(\lim_{\mathcal{U}}(x_\alpha, z))(\lim_{\mathcal{U}}(y, x_\alpha)) = 0$ for all y and z in $\mathcal{H}$. The lemma is proved.

(5.3) THEOREM. *If $\{x_\alpha\}$ is a set of unit vectors in $\mathcal{H}$, indexed over a set κ, which is norm dense in the set of all unit vectors in $\mathcal{H}$, then $\Lambda[x_\alpha]$ maps $\beta\kappa$ onto $\mathcal{S}(\mathcal{B}(\mathcal{H}))$, the pure state space of $\mathcal{B}(\mathcal{H})$.*

PROOF. If A is a selfadjoint operator which is not a positive operator, then $(Ax, x) < 0$ for some unit vector x in $\mathcal{H}$. Since the x_α's are dense in the set of unit vectors, $(Ax_\alpha, x_\alpha) < 0$ for some index α, so $\{x_\alpha\}$ is total for $\mathcal{B}(\mathcal{H})$. Hence, by (4.3), $\mathcal{S}(\mathcal{B}(\mathcal{H}))$ is contained in the range of $\Lambda[x_\alpha]$. To show the reverse inclusion, let $\mathcal{U}$ be an ultrafilter on κ; then $x_\alpha \rightharpoonup y(\mathcal{U})$ for some y in $\mathcal{H}$ with $\|y\| \leq 1$ as we noted in (5.1). If $\|y\| = 1$, then $x_\alpha \to y(\mathcal{U})$ and $\Lambda_{\mathcal{U}}[x_\alpha] = \omega_y$ so that $\Lambda_{\mathcal{U}}[x_\alpha] \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$. If $y = 0$, $\Lambda_{\mathcal{U}}[x_\alpha]$ is a singular state by (5.2) and, again, $\Lambda_{\mathcal{U}}[x_\alpha] \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$. So suppose that $\|y\|^2 = t$ where $0 < t < 1$. Let $z = t^{-1/2}y$ and for each α in κ let $z_\alpha = (x_\alpha - y)\|x_\alpha - y\|^{-1}$. Then $z_\alpha \rightharpoonup O(\mathcal{U})$ so the state $f = t\omega_z + (1 - t)\Lambda_{\mathcal{U}}[z_\alpha]$ is in $\mathcal{S}(\mathcal{B}(\mathcal{H}))$. We show that $f = \Lambda_{\mathcal{U}}[x_\alpha]$. Indeed, if T is an operator, then

$$\begin{aligned}\Lambda_{\mathcal{U}}[x_\alpha](T) &= \lim_{\mathcal{U}} (Tx_\alpha, x_\alpha)\\ &= f + \lim_{\mathcal{U}} (Ty, z_\alpha)\|x_\alpha - y\| + \lim_{\mathcal{U}} (Tz_\alpha, y)\|x_\alpha - y\|,\end{aligned}$$

because $\lim_{\mathcal{U}}\|x_\alpha - y\|^2 = 1 - t$. Since $z_\alpha \rightharpoonup O(\mathcal{U})$, $\lim_{\mathcal{U}}(Ty, z_\alpha) = \lim_{\mathcal{U}}(z_\alpha, T^*y) = 0$. Hence, $\Lambda_{\mathcal{U}}[x_\alpha] = f$ and the theorem is proved.

Of course, the map $\Lambda[x_\alpha]$ is not one-to-one since for each unit vector y in $\mathcal{H}$, there are distinct ultrafilters $\mathcal{U}$ and $\mathcal{V}$ so that

$$\lim_{\mathcal{U}} \|x_\alpha - y\| = \lim_{\mathcal{V}} \|x_\alpha - y\| = 0.$$

One might expect, however, that if σ and τ are subsets of κ such that $\inf\{\|x_\alpha - x_\beta\|: \alpha \in \sigma, \beta \in \tau\} > 0$ and $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters such that $\sigma \in \mathcal{U}$ and $\tau \in \mathcal{V}$, then $\Lambda_{\mathcal{U}}[x_\alpha] \neq \Lambda_{\mathcal{V}}[x_\alpha]$. Our next results show that this is not the case.

(5.4) Henceforth, we shall assume that $\mathcal{H}$ is a separable complex Hilbert space with a fixed (infinite) orthonormal basis $\{e_n\}$. Let $\mathcal{D}$ be the diagonal operators in $\{e_n\}$. (Thus, an operator D is in $\mathcal{D}$ if and only if each e_n is an eigenvector for D.) We shall show that for each free ultrafilter $\mathcal{U}$ on $\mathbf{N}$ there are many sequences $\{x_m\}$ of unit vectors such that $\inf_{n,m}\|x_m - e_n\| > 0$, but $\Lambda_{\mathcal{U}}[e_n] = \Lambda_{\mathcal{V}}[x_m]$ for some ultrafilter $\mathcal{V}$ on $\mathbf{N}$. Thus, the map $\Lambda[x_\alpha]$ of the previous paragraph is noninjective in a "nontrivial way". Also our results give an indication of the form which a sequence $\{x_m\}$ of unit vectors must have in order that $\Lambda_{\mathcal{U}}[e_n]$ agree with $\Lambda_{\mathcal{V}}[x_m]$ on $\mathfrak{D}$, but disagree on $\mathfrak{B}(\mathcal{H})$.

(5.5) Let $(\mathbf{N} \times \mathbf{N})^*$ denote all ordered pairs of integers (i, j) such that $i \neq j$ and let θ be a one-to-one map of $(\mathbf{N} \times \mathbf{N})^*$ onto $\mathbf{N}$. Let a and b be complex numbers such that $0 < |a| < 1$ and $|a|^2 + |b|^2 = 1$. For each integer m let $i(m)$ and $j(m)$ be the first and second coordinates of $\theta^{-1}(m)$, respectively, and let $x_m = ae_{i(m)} + be_{j(m)}$. For each subset σ of $\mathbf{N}$ let P_σ be the projection of $\mathcal{H}$ onto $\mathrm{sp}\{e_n \colon n \in \sigma\}$.

THEOREM. *If $\mathcal{U}$ is a free ultrafilter on* $\mathbf{N}$, *then there is a free ultrafilter $\mathcal{V}$ on* $\mathbf{N}$ *so that $\Lambda_{\mathcal{U}}[e_n]$ and $\Lambda_{\mathcal{V}}[x_m]$ agree on $\mathfrak{D}$, where $\{x_m\}$ is as above.*

PROOF. Let $\mathcal{I}$ be the ideal of compact operators in $\mathfrak{D}$ and let $\mathcal{C}$ be the C^*-algebra $\mathfrak{D}/\mathcal{I}$. View $\Lambda[x_m]$ as mapping $\beta\mathbf{N}$ into $\mathfrak{S}(\mathfrak{D})$ and let Ξ be those ultrafilters $\mathcal{V}$ on $\mathbf{N}$ such that $x_m \rightharpoonup O(\mathcal{V})$. Then $\Xi \subset \beta\mathbf{N} - \mathbf{N}$, and for each $\mathcal{V}$ in Ξ, $\Lambda_{\mathcal{V}}[x_m]$ is a singular state (on $\mathfrak{B}(\mathcal{H})$). In particular, $\Lambda_{\mathcal{V}}[x_m](\mathcal{I}) \equiv 0$. Hence, $\mathfrak{R}' = \{\Lambda_{\mathcal{V}}[x_m] \colon \mathcal{V} \in \Xi\}$ may be regarded a subset of $\mathfrak{S}(\mathcal{C})$. We show that $\mathfrak{R}'$ is total for $\mathcal{C}$. Let π be the natural map of $\mathfrak{D}$ onto $\mathcal{C}$. Recall that $\pi(D) = \pi(D)^*$ in $\mathcal{C}$ if and only if $\pi(D) = \pi(A)$ for some selfadjoint element A in $\mathfrak{D}$. Hence, if $\pi(A)$ is a selfadjoint element of $\mathcal{C}$ which is not positive in $\mathcal{C}$, we may assume that A is a selfadjoint element of $\mathfrak{D}$. Since $\pi(A)$ is not positive, for some $\varepsilon > 0$, $\sigma_\varepsilon = \{n \colon (Ae_n, e_n) < -\varepsilon\}$ is infinite. Let $\tau = \{m \colon i(m) \in \sigma_\varepsilon, j(m) \in \sigma_\varepsilon\}$. Clearly, there is a free ultrafilter $\mathcal{V}$ on $\mathbf{N}$ so that $\tau \in \mathcal{V}$ and $x_m \rightharpoonup O(\mathcal{V})$. Then $\Lambda_{\mathcal{V}}[x_m](A) = \Lambda_{\mathcal{V}}[x_m](\pi(A)) < -\varepsilon$. Since $\mathcal{V} \in \Xi$, $\mathfrak{R}'$ is total for $\mathcal{C}$. Since the range of $\Lambda[x_m]$ restricted to Ξ is compact in $\mathfrak{S}(\mathfrak{D})$, it is compact in $\mathfrak{S}(\mathcal{C})$. Hence, by [7, 3.4.1] $\mathfrak{R}'$ contains the pure states (= the pure state space) of $\mathcal{C}$. Since $\Lambda_{\mathcal{U}}[e_n]$ is a pure state on $\mathcal{C}$, the theorem is proved.

(5.6) THEOREM. *If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters such that $\Lambda_{\mathcal{U}}[e_n]$ and $\Lambda_{\mathcal{V}}[x_m]$ agree on $\mathfrak{D}$ (where $\{x_m\}$ is as in* (5.5)) *then $\Lambda_{\mathcal{U}}[e_n] = \Lambda_{\mathcal{V}}[x_m]$.*

PROOF. Let $\mu_1 = \{m \colon i(m) < j(m)\}$ and $\mu_2 = \{m \colon i(m) > j(m)\}$. Then $\mu_1 \cup \mu_2 = \mathbf{N}$ so either $\mu_1 \in \mathcal{V}$ or $\mu_2 \in \mathcal{V}$. Since the proof is the same in either case, assume $\mu_1 \in \mathcal{V}$. For each operator T,

$$\begin{aligned}\Lambda_{\mathcal{V}}[x_m](T) &= |a|^2 \lim_{\mathcal{V}} (Te_{i(m)}, e_{i(m)}) + ab^* \lim_{\mathcal{V}} (Te_{i(m)}, e_{j(m)}) \\ &\quad + a^* b \lim_{\mathcal{V}} (Te_{j(m)}, e_{i(m)}) + |b|^2 \lim_{\mathcal{V}} (Te_{j(m)}, e_{j(m)}) \\ &= |a|^2 f_1(T) + ab^* \xi_1(T) + a^* b \xi_2(T) + |b|^2 f_2(T),\end{aligned}$$

where f_1 and f_2 are the states and ξ_1 and ξ_2 are the linear functionals which this decomposition determines. Note that since $i(m) \neq j(m)$ for all m, $\xi_1(D) = \xi_2(D) = 0$ for all D in $\mathfrak{D}$. Thus, on $\mathfrak{D}$, $\Lambda_{\mathfrak{U}}[e_n] = \Lambda_{\mathfrak{V}}[x_m] = |a|^2 f_1 + |b|^2 f_2$, and, since $\Lambda_{\mathfrak{U}}[e_n]$ is a pure state on $\mathfrak{D}$, $f_1 = f_2 = \Lambda_{\mathfrak{U}}[e_n]$ on $\mathfrak{B}(\mathfrak{H})$. Also, for each operator T, $\xi_2(T) = (\xi_1(T^*))^*$. Thus, to prove the theorem, it suffices to show that $\xi_1 \equiv 0$. Assume that for some operator T with $\|T\| = 1$, $\xi_1(T) = \lambda \neq 0$. We will show that this assumption implies that $\Lambda_{\mathfrak{V}}[x_m]$ is not a homomorphism on $\mathfrak{D}$, a contradiction. For each integer i, let $\rho_i = \{j: |(Te_i, e_j)| > \frac{1}{2}|\lambda|\}$ and let $n_0 = \min\{n: \frac{1}{4}n|\lambda|^2 \geqslant 1\}$. Then, since $\|Te_i\| \leqslant \|T\| = 1$, the cardinality of each ρ_i is no greater than n_0. Let $\tau = (\bigcup_i\{m: j(m) \in \rho_i\}) \cap \mu_1$. Then $\tau \in \mathfrak{V}$, for otherwise $|\xi_1(T)| \leqslant \frac{1}{2}|\lambda|$. Partition τ into at most n_0 subsets so that if k and m both belong to the same subset, $i(k) \neq i(m)$. Since exactly one of these subsets is in $\mathfrak{V}$, we may assume that τ enjoys this property. For each integer j the set $\sigma_j = \{i: |(Te_i, e_j)| > \frac{1}{2}|\lambda|\}$ has cardinality at most n_0 because $\|T^*e_j\| \leqslant \|T\| = 1$. Hence, arguing as above, we may assume that if k and m are distinct integers in τ, then $i(k) \neq i(m)$ and $j(k) \neq j(m)$. Since $\tau \subset \mu_1$, for each m in τ there is at most one integer k in τ so that $j(k) = i(m)$ and at most one integer p so that $j(m) = i(p)$. Thus, τ can be partitioned in a natural way into linearly ordered chains such that if m is in a given chain, its unique successor (if it has one) is that element p in τ such that $j(m) = i(p)$ and such that if k and m belong to distinct chains, then $\{i(k), j(k)\} \cap \{i(m), j(m)\} = \varnothing$. Form τ_1 by choosing every other element of each chain, and let $\tau_2 = \tau - \tau_1$. Then either $\tau_1 \in \mathfrak{V}$ or $\tau_2 \in \mathfrak{V}$. Since the proof is the same in either case, assume $\tau_1 \in \mathfrak{V}$. Then if k and m are distinct elements of τ_1, $\{i(k), j(k)\} \cap \{i(m), j(m)\} = \varnothing$. Let $\sigma = \{i(m): m \in \tau_1\}$. Then $(P_\sigma x_m, x_m) = |a|^2$ for all m in τ_1 since $i(m) \in \sigma$, but $j(m) \notin \sigma$. Hence, $\Lambda_{\mathfrak{V}}[x_m](P_\sigma) = |a|^2$ so $\Lambda_{\mathfrak{V}}[x_m]$ is not a homomorphism on $\mathfrak{D}$ and the theorem is proved.

(5.7) For each x in $\mathfrak{H}$, let $\#(x)$ be the cardinality of $\{n: (x, e_n) \neq 0\}$, where $\{e_n\}$ is a fixed orthonormal basis for $\mathfrak{H}$ as before.

THEOREM. *If $\{x_m\}$ is a sequence of unit vectors in $\mathfrak{H}$ such that $\sup_m \#(x_m) < \infty$ and $\mathfrak{U}$ and $\mathfrak{V}$ are ultrafilters on* **N** *such that $\Lambda_{\mathfrak{V}}[x_m]$ agrees with $\Lambda_{\mathfrak{U}}[e_n]$ on $\mathfrak{D}$, then $\Lambda_{\mathfrak{V}}[x_m] = \Lambda_{\mathfrak{U}}[e_n]$.*

PROOF. In order to simplify notation we only consider the case $\sup_m \#(x_m) = 3$. The proof in other cases is similar. So assume $x_m = a_m e_{i(m)} + b_m e_{j(m)} + c_m e_{k(m)}$, where $|a_m|^2 + |b_m|^2 + |c_m|^2 = 1$ and $i(m), j(m)$ and $k(m)$ are distinct for each m. Let $a = \lim_{\mathfrak{V}} a_m$, $b = \lim_{\mathfrak{V}} b_m$ and $c = \lim_{\mathfrak{V}} c_m$, and let $x'_m = ae_{i(m)} + be_{j(m)} + ce_{k(m)}$. Then $\|x'_m\| = 1$ and $\lim_{\mathfrak{V}}\|x_m - x'_m\| = 0$, so $\Lambda_{\mathfrak{V}}[x_m] = \Lambda_{\mathfrak{V}}[x'_m]$. Thus, $\Lambda_{\mathfrak{V}}[x_m] = |a|^2 f_1 + |b|^2 f_2 + |c|^2 f_3 + \xi$, where $f_1 = \Lambda_{\mathfrak{V}}[e_{i(m)}]$, $f_2 = \Lambda_{\mathfrak{V}}[e_{j(m)}]$, $f_3 = \Lambda_{\mathfrak{V}}[e_{k(m)}]$ and ξ is the linear functional comprised of the six cross-terms. That is, $\xi = \xi_1 + \xi_2 + \cdots + \xi_6$ where, for example, $\xi_1(T) = ab^* \lim_{\mathcal{V}}(Te_{i(m)}, e_{j(m)})$ and $\xi_2(T) = a^*b \lim_{\mathcal{V}}(Te_{j(m)}, e_{i(m)})$. Since $i(m), j(m)$ and $k(m)$ are distinct for each m,

$$\Lambda_{\mathcal{U}}[e_n](D) = \Lambda_{\mathcal{V}}[x_m](D) = |a|^2 f_1(D) + |b|^2 f_2(D) + |c|^2 f_3(D)$$

for each D in $\mathcal{D}$ and, as in the first part of the proof of (5.6), $f_1 = f_2 = f_3 = \Lambda_{\mathcal{U}}[e_n]$. Suppose that $|c| \neq 1$ and let $g = (1 - |c|^2)^{-1}(|a|^2 f_1 + \xi_1 + \xi_2 + |b|^2 f_2)$. Then g is a state on $\mathcal{B}(\mathcal{H})$ and g agrees with $\Lambda_{\mathcal{U}}[e_n]$ on $\mathcal{D}$. Furthermore, g has the form considered in (5.6). Hence, $g = \Lambda_{\mathcal{U}}[e_n]$ by (5.6). Thus, $\xi_1 + \xi_2 \equiv 0$. Similarly, $\xi_3 + \xi_4 \equiv \xi_5 + \xi_6 \equiv 0$ so $\xi \equiv 0$ and $\Lambda_{\mathcal{U}}[e_n] = \Lambda_{\mathcal{V}}[x_m]$.

6. States on $\mathcal{W}(n_j)$.

(6.1) We continue to assume that $\{e_n\}$ is a fixed orthonormal basis for $\mathcal{H}$. Let $\{n_j\}$ be a sequence of distinct integers, let $n_0 = 0$ and let $d_j = n_j - n_{j-1}$ for $j = 1, 2 \ldots.$ Let $\mathcal{H}_j = \text{sp}\{e_n\colon n_{j-1} < n \leqslant n_j\}$ for $j = 1, 2, \ldots.$ Then $\{\mathcal{H}_j\}$ is a family of mutually orthogonal subspaces of $\mathcal{H}$ such that the dimension of $\mathcal{H}_j$ is d_j and $\mathcal{H} = \Sigma \oplus \mathcal{H}_j$. We define $\mathcal{W}(n_j)$ to be $\Sigma \oplus \mathcal{B}(\mathcal{H}_j)$. Thus, a typical element of $\mathcal{W}(n_j)$ has the form $T = \Sigma \oplus T_j$ where T_j acts on $\mathcal{H}_j$ and $\|T\| = \sup_j \|T_j\|$. A projection in $\mathcal{W}(n_j)$ is abelian if and only if it has the form $\Sigma z_j \otimes z_j$ where for each j, $z_j \in \mathcal{H}_j$ and $\|z_j\|$ is 0 or 1. ($z \otimes z$ is the operator defined in the proof of (5.2).) Each ultrafilter $\mathcal{U}$ on $\mathbf{N}$ induces a trace $\text{tr}_{\mathcal{U}}$ on $\mathcal{W}(n_j)$ as follows. Let tr_j be the canonical trace on $\mathcal{B}(\mathcal{H}_j)$ normalized so that $\text{tr}_j(I_j) = 1$, where I_j is the identity on $\mathcal{H}_j$. Let P_j be the projection of $\mathcal{H}$ onto $\mathcal{H}_j$, and for each T in $\mathcal{W}(n_j)$ put $\text{tr}_{\mathcal{U}}(T) = \lim_{\mathcal{U}} \text{tr}_j(P_j T)$.

In this section we show that many states on $\mathcal{W}(n_j)$ have the form $\Lambda_{\mathcal{U}}[x_n]$ where each $x_n \in \mathcal{H}_j$ for some j and $\{n\colon x_n \in \mathcal{H}_j\}$ is finite. For example, if $\mathcal{U}$ is a free ultrafilter on $\mathbf{N}$ and $d_j \to \infty$, $\text{tr}_{\mathcal{U}}$ has this form. Also, we show that if f is a state on $\mathcal{W}(n_j)$ such that $f(P_j) = 0$ for $j = 1, 2\ldots.$ and such that f is a homomorphism on $\mathcal{D}$ (the diagonal operators in the basis $\{e_n\}$), then f has this form. This is of interest since it follows from (8.2) that a homomorphism of $\mathcal{D}$ has a unique state extension to $\mathcal{B}(\mathcal{H})$ if (and only if) it has a unique state extension to $\mathcal{W}(n_j)$ for all sequences $\{n_j\}$.

(6.2) Let $\{x_n\}$ be a sequence of unit vectors such that each $x_n \in \mathcal{H}_j$ for some j, $\{n\colon x_n \in \mathcal{H}_j\}$ is finite for each j and for each unit vector y in $\mathcal{H}_j$ there is an n so that $\|x_n - y\| < 1/j$.

THEOREM. *If f is a state in $\mathcal{S}(\mathcal{W}(n_j))$ such that $f(P_j) = 0$ for each j, then $f = \Lambda_{\mathcal{U}}[x_n]$ for some free ultrafilter $\mathcal{U}$ on* $\mathbf{N}$.

PROOF. The proof is similar to that of (5.5) so we omit the details. As in (5.5) one shows that the restricted range $\mathcal{R}'$ of $\Lambda[x_m]$ is total for $\mathcal{C}$ where now $\mathcal{C}$ is $\mathcal{W}(n_j)$ modulo the compact operators in $\mathcal{W}(n_j)$. The theorem then follows by [7, 3.4.1].

(6.3) COROLLARY. *If* $\lim_j d_j = \infty$ *and* $\mathcal{U}$ *is a free ultrafilter on* $\mathbf{N}$ *then there is a free ultrafilter* $\mathcal{V}$ *on* $\mathbf{N}$ *so that* $\mathrm{tr}_{\mathcal{U}} = \Lambda_{\mathcal{V}}[x_n]$.

PROOF. The center $\mathfrak{Z}$ of $\mathcal{W}(n_j)$ is generated as a von Neumann algebra by $\{P_j\}$. Hence, $\mathrm{tr}_{\mathcal{U}}$ is a homomorphism on $\mathfrak{Z}$. Furthermore, for any unit vector z in $\mathcal{H}_j$ $\mathrm{tr}_j(z \otimes z) = d_j^{-1}$. Thus, since $\mathcal{U}$ is a free ultrafilter and $d_j \to \infty$, $\mathrm{tr}_{\mathcal{U}}(E) = 0$ for all abelian projections E in $\mathcal{W}(n_j)$. So by Glimm's theorem $\mathrm{tr}_{\mathcal{U}} \in \mathcal{S}(\mathcal{W}(n_j))$ and, since $\mathrm{tr}_{\mathcal{U}}(P_j) = 0$ for all j, $\mathrm{tr}_{\mathcal{U}}$ has the desired form by (6.2).

(6.4) THEOREM. *If f is a state on $\mathcal{W}(n_j)$ such that f is a homomorphism on $\mathfrak{D}$, then either $f(P) = 1$ for some abelian projection P in $\mathfrak{D}$ or else $f(E) = 0$ for all abelian projections E in $\mathcal{W}(n_j)$. In particular, if $f(P_j) = 0$ for all j then $f = \Lambda_{\mathcal{V}}[x_n]$ for some free ultrafilter $\mathcal{V}$ on* $\mathbf{N}$.

PROOF. By our hypothesis, f agrees with $\Lambda_{\mathcal{U}}[e_n]$ on $\mathfrak{D}$ for some ultrafilter $\mathcal{U}$ on $\mathbf{N}$. Assume that $f(P) = 0$ for each abelian projection P in $\mathfrak{D}$ and assume that $f(E) = a > 0$ for some abelian projection E in $\mathcal{W}(n_j)$. We show that these assumptions lead to a contradiction. As remarked in (6.1) $E = \Sigma z_j \otimes z_j$ where $z_j \in \mathcal{H}_j$ and $\|z_j\|$ is 0 or 1 for each j. For each pair of integers (j, k) let $\tau_{jk} = \{n: |(z_j, e_n)|^2 > k^{-1}\}$ and let $\tau_k = \bigcup_j \tau_{jk}$. Then the cardinality of each τ_{jk} is at most k since $\|z_j\| \leqslant 1$. Hence, the projection P_{τ_k} (onto $\mathrm{sp}\{e_n: n \in \tau_k\}$) is the sum of at most k abelian projections in $\mathfrak{D}$ and so $f(P_{\tau_k}) = 0$ for each integer k. Thus, $P_{\tau_k}^{\perp} \in \mathcal{G}_f$ and so $f(A_k) = a$, where $A_k = P_{\tau_k}^{\perp} E P_{\tau_k}^{\perp}$. Let $y_{jk} = P_{\tau_k}^{\perp} z_j$ for each pair of integers j and k. For each subset ρ of $\mathbf{N}$ let $\sigma(\rho) = \bigcup_{j \in \rho}\{n: n_{j-1} < n \leqslant n_j\}$. For each integer k, let $\rho_k = \{j: \|y_{jk}\| > \frac{1}{2}a\}$. Then $\|P_{\sigma(\rho_k)}^{\perp} A_k P_{\sigma(\rho_k)}^{\perp}\| \leqslant \frac{1}{2}a$, so $\sigma(\rho_k) \in \mathcal{U}$ for each k. Choose $k > 16a^{-2}$. Then if $j \in \rho_k$, $|(y_{jk}, e_n)|^2 < 16^{-1}a^2$ for each n and $\Sigma_n |(y_{jk}, e_n)|^2 > 4^{-1}a^2$. Hence for each j in ρ_k we may partition the interval $\{n_{j-1} < n \leqslant n_j\}$ into r subintervals $\mu_{j1}, \mu_{j2}, \ldots, \mu_{jr}$ so that $\Sigma_{n \in \mu_{ji}} |(y_{jk}, e_n)|^2 < 8^{-1}a^2$ for $1 \leqslant i \leqslant r$, where r does not depend on j. (Some of the μ_{ji}'s may be empty.) Let $\mu_i = \bigcup_j \mu_{ji}$ for $1 \leqslant i \leqslant r$. Then $\mu_1 \cup \cdots \cup \mu_r = \sigma(\rho_k)$, so $\mu_i \in \mathcal{U}$ for exactly one i. Say $\mu_1 \in \mathcal{U}$. Then $\|P_{\mu_1} A_k P_{\mu_1}\| \leqslant 8^{-1/2}a$, but $f(P_{\mu_1} A_k P_{\mu_1}) = a$, a contradiction. This proves our first assertion. Since $\mathfrak{D}$ contains the center $\mathfrak{Z}$ of $\mathcal{W}(n_j)$, f is a homomorphism on $\mathfrak{Z}$. Thus, by our first assertion and Glimm's theorem, $f \in \mathcal{S}(\mathcal{W}(n_j))$. Hence, if $f(P_j) = 0$ for all j, f has the desired form by (6.2).

7. States on $\mathcal{C}$.

(7.1) Let $\mathcal{H} = L^2(0, 1)$, the square integrable functions on the real interval $[0, 1]$ with Lebesgue measure λ. Let $L^\infty(0, 1)$ be the essentially bounded functions on the same measure space. Then each φ in $L^\infty(0, 1)$ determines an operator M_φ on $L^2(0, 1)$ by the formula $M_\varphi f = \varphi f$ for f in $L^2(0, 1)$. Let $\mathcal{C} = \{M_\varphi: \varphi \in L^\infty(0, 1)\}$. Then $\mathcal{C}$ is a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$.

In **[10]** Kadison and Singer showed that there are many disinct conditional expectations of $\mathfrak{B}(\mathfrak{H})$ onto $\mathcal{C}$ and used this fact to infer that not all homomorphisms of $\mathcal{C}$ have unique state extensions to $\mathfrak{B}(\mathfrak{H})$. In this section we show that the orthonormal basis $\{\varphi_n\}$ formed by the Haar functions is total for $\mathcal{C}$. It follows from this fact and some of our previous results that

(i) each homomorphism h on $\mathcal{C}_h$ as distinct state extensions to $\mathfrak{B}(\mathfrak{H})$, and

(ii) for each homomorphism h on $\mathcal{C}$ there is a pure state f on $\mathfrak{B}(\mathfrak{H})$ such that f agrees with h on $\mathcal{C}$ and f is a homomorphism on $\mathfrak{D}$ (the diagonal operators in the basis $\{\varphi_n\}$).

(7.2) We recall the definition of the Haar functions. For each integer j and each integer $i \leqslant 2^j$ let $\chi_{i,j}$ be the characteristic function of the interval $[(i-1)/2^j, i/2^j]$. The *Haar functions* $\{\varphi_n\}$ are defined as follows. Let

$$\varphi_1 \equiv 1, \quad \varphi_2 = \chi_{1,1} - \chi_{2,1}, \quad \varphi_{i+2^j} = (2^{1/2})^j(\chi_{2i-1,j+1} - \chi_{2i,j+1})$$

for $j = 1, 2, \dots$ and $1 \leqslant i \leqslant 2^j$. Clearly, the system $\{\varphi_n\}$ is an orthonormal subset of $L^2(0, 1)$. In fact, the Haar functions form an orthonormal basis for $L^2(0, 1)$ **[12**, p. 49].

If δ is a Borel subset of $[0, 1]$, let P_δ be the projection in $\mathcal{C}$ induced by χ_δ, the characteristic function of δ.

THEOREM. *If h is a nonzero homomorphism of $\mathcal{C}$, then there is a free ultrafilter $\mathfrak{U}$ on* $\mathbf{N}$ *such that h and $\Lambda_{\mathfrak{U}}[\varphi_n]$ agree on $\mathcal{C}$, where $\{\varphi_n\}$ denotes the Haar functions.*

PROOF. By (4.3) it suffices to show that $\{\varphi_n\}$ is total for $\mathcal{C}$. Let M_φ be a selfadjoint element of norm 1 in $\mathcal{C}$ which is not positive. Then φ is not essentially positive; that is, there is a Borel set δ in $[0, 1]$ so that $\lambda(\delta) = a > 0$ and for some $b > 0$, $\varphi(t) < -b$ for all t in δ. Choose an open set V in $[0, 1]$ so that $\delta \subset V$ and $\lambda(V - \delta) < \frac{1}{2}ab$. Note that for each $n > 2$, $\varphi_n^2 = 2^j\chi_{i,j}$ where $n = i + 2^j$. It follows that $\chi_V = \sum_{n=1}^{\infty} d_n\varphi_n^2$ where $0 \leqslant d_n$ for each n. Hence,

$$\sum d_n \int \varphi\varphi_n^2\, d\lambda = \int \varphi\chi_V\, d\lambda = \int_\delta \varphi\, d\lambda + \int_{V-\delta} \varphi\, d\lambda$$
$$\leqslant -ab + \tfrac{1}{2}ab\|\varphi\| = -\tfrac{1}{2}ab < 0.$$

Thus, for at least one n, $(M_\varphi\varphi_n, \varphi_n) = \int \varphi\varphi_n^2\, d\lambda < 0$ so the Haar functions are total for $\mathcal{C}$ and the theorem is proved.

(7.3) COROLLARY. *If h is a nonzero homomorphism on $\mathcal{C}$, then there is a pure state f on $\mathfrak{B}(\mathfrak{H})$ such that f agrees with h on $\mathcal{C}$ and f is also a homomorphism on $\mathfrak{D}$, the diagonal operators determined by the Haar functions.*

PROOF. Let $\mathcal{A}$ be the C^*-algebra generated by $\mathcal{C}$ and $\mathfrak{D}$. If h is a homomorphism on $\mathcal{C}$ then by (7.2) there is an ultrafilter $\mathfrak{U}$ on $\mathbf{N}$ so that $\Lambda_{\mathfrak{U}}[\varphi_n]$ agrees with h on $\mathcal{C}$. Then $\Lambda_{\mathfrak{U}}[\varphi_n]$ is a homomorphism on $\mathcal{A}$ and so there is a pure state f on $\mathfrak{B}(\mathcal{H})$ which agrees with $\Lambda_{\mathfrak{U}}[\varphi_n]$ on $\mathcal{A}$.

(7.4) THEOREM. *If h is a homomorphism on $\mathcal{C}$ then there are distinct pure states f and g on $\mathfrak{B}(\mathcal{H})$ such that f and g both agree with h on $\mathcal{C}$.*

PROOF. Define a unitary operator S on $\mathcal{H}$ as follows. Let $S\varphi_1 = \varphi_2$, $S\varphi_2 = \varphi_1$, $S\varphi_n = \varphi_{n+1}$ if $n \neq 2^j$ for all j, $S\varphi_{2^{j+1}} = \varphi_{1+2^j}$ for $j = 1, 2, \ldots$. Then S permutes the Haar functions and so is a unitary operator on $\mathcal{H}$. Note that $(S\varphi_n, \varphi_n) = 0$ for all integers n. Let δ be a Borel set with $\lambda(\delta) > 0$. We show that $\|P_\delta S P_\delta\| = 1$. Since $P_\delta \neq 0$, there is a homomorphism h of $\mathcal{C}$ such that $h(P_\delta) = 1$. Hence, by (7.2) there is a free ultrafilter $\mathfrak{U}$ on $\mathbf{N}$ so that $\Lambda_{\mathfrak{U}}[\varphi_n](P_\delta) = 1$. Fix $\varepsilon > 0$ and choose an integer $n = i + 2^j$ so that

$$(P_\delta\varphi_n, \varphi_n) = \int_\delta \varphi_n^2\, d\lambda > 1 - \varepsilon.$$

Since $\varphi_n^2 = 2^j\chi_{i,j}$,

$$\lambda\big(\big[(i-1)/2^j, i/2^j\big] \cap \delta\big) > \left(\tfrac{1}{2}\right)^j(1-\varepsilon).$$

Let

$$\alpha = \lambda\big(\big[2(i-1)/2^{j+1}, (2i-1)/2^{j+1}\big] \cap \delta\big).$$

Then $\alpha + (\frac{1}{2})^{j+1} > (\frac{1}{2})^j(1-\varepsilon)$, so $\alpha > (\frac{1}{2})^{j+1}(1-2\varepsilon)$. Similarly,

$$\lambda\big(\big[(2i-1)/2^{j+1}, 2i/2^{j+1}\big] \cap \delta\big) > \left(\tfrac{1}{2}\right)^{j+1}(1-2\varepsilon).$$

Let $m = 2i - 1 + 2^{j+1}$. Then φ_m and φ_{m+1} are the Haar functions "under" φ_n and $S\varphi_m = \varphi_{m+1}$. Furthermore,

$$\|P_\delta\varphi_m\|^2 = \int_\delta \varphi_m^2 = 2^{j+1}\alpha > 1 - 2\varepsilon \quad \text{and} \quad \|P_\delta\varphi_{m+1}\|^2 > 1 - 2\varepsilon.$$

Thus,

$$\begin{aligned}\|P_\delta S P_\delta \varphi_m\| &\geq \|P_\delta S\varphi_m\| - \|P_\delta S(\varphi_m - P_\delta\varphi_m)\| \\ &\geq \|P_\delta\varphi_{m+1}\| - \|\varphi_m - P_\delta\varphi_m\| > (1-2\varepsilon)^{1/2} - (2\varepsilon)^{1/2}.\end{aligned}$$

Since ε was arbitrary $\|P_\delta S P_\delta\| = 1$. Let h be a homomorphism on $\mathcal{C}$. Assume that h has a unique state extension f_h to $\mathfrak{B}(\mathcal{H})$. Then by (7.2), $f_h = \Lambda_{\mathfrak{U}}[\varphi_n]$ for some free ultrafilter $\mathfrak{U}$ on $\mathcal{H}$. Hence, $f_h(S) = 0$ and by (3.2) there is a projection P_δ in $\mathcal{C}$ so that $h(P_\delta) = 1$ and $\|P_\delta S P_\delta\| < \frac{1}{2}$. Since this contradicts the first part of the proof, h cannot have a unique state extension to $\mathfrak{B}(\mathcal{H})$.

(7.5) If $\{\psi_n\}$ is a sequence of unit vectors in $L^2(0, 1)$ with pairwise disjoint support, then it is easy to construct a Borel set δ in $[0, 1]$ so that $(P_\delta\psi_n, \psi_n) = \frac{1}{2}$ for all n. Thus, $\Lambda_{\mathfrak{U}}[\psi_n]$ is not a homomorphism on $\mathcal{C}$ for any ultrafilter $\mathcal{U}$. Our next result shows that if for some sequence of unit vectors $\{\psi_n\}$ and some ultrafilter $\mathcal{U}$, $\Lambda_{\mathcal{U}}[\psi_n]$ is a homomorphism, then the supports of the ψ_n's must "overlap infinitely often".

Recall that if h is a homomorphism on $\mathcal{C}$, then there is a unique real number a in $[0, 1]$ such that for each open subset V in $[0, 1]$, $h(P_V) = 1$ if and only if $a \in V$. In fact $a = \bigcap \{V\colon V$ is an open interval and $h(P_V) = 1\}$. We describe this circumstance by saying *h is in the fiber over a.*

THEOREM. *Let $\{\psi_n\}$ be a sequence of unit vectors in $L^2(0, 1)$ and let $\mathcal{U}$ be an ultrafilter on $\mathbf{N}$ so that $\Lambda_{\mathcal{U}}[\psi_n]$ agrees with a homomorphism h on $\mathcal{C}$. If h is in the fiber over a and $\{a_n\}$ is a sequence in $[0, 1]$ such that $\lim_n a_n = a$, then $\lim_{\mathcal{U}} \int_a^{a_n} |\psi_n|^2\, d\lambda = 0$.*

PROOF. Let $\lim_{\mathcal{U}} \int_a^{a_n} |\psi_n|^2\, d\lambda = r$. We show that if $r > 0$ then $\Lambda_{\mathcal{U}}[\psi_n]$ is not a homomorphism on $\mathcal{C}$. Hence assume $r > 0$. Then either $\{n\colon a < a_n < 1\} \in \mathcal{U}$ or $\{n\colon 0 < a_n < a\} \in \mathcal{U}$. Since the proof is similar in either case we shall assume that $a < a_n < 1$ for all n. Also, passing to (another) subsequence if necessary we may assume that $\int_a^{a_n} |\psi_n|^2\, d\lambda \geq 3r/4$ for each integer n. Choose b_n for each n so that $a < b_n < a_n$ and $\int_a^{b_n} |\psi_n|^2\, d\lambda = \frac{1}{2} r$. We define a sequence of integers $\{n_k\}$ and a sequence of real numbers $\{c_k\}$ inductively as follows. Let $c_1 = b_1$ and $n_1 = 1$. If $n_1 < n_2 < \cdots < n_k$ and $c_1, c_2, \ldots, c_k$ have been chosen, pick n_{k+1} so that $n_{k+1} > n_k$ and so that if $n > n_{k+1}$, then $a_n < c_k$. Let $c_{k+1} = \min\{b_n\colon n_k < n \leq n_{k+1}\}$. Note that if $k > 1$ and $n_k < n \leq n_{k+1}$, then $c_{k+1} \leq b_n < a_n < c_{k-1}$. Thus, $c_1 > c_3 > \ldots$, and $c_2 > c_4 > \ldots$. Let $\sigma = \{n\colon n_k < n \leq n_{k+1}, k$ an odd integer$\}$. Then either $\sigma \in \mathcal{U}$ or $\mathbf{N} - \sigma \in \mathcal{U}$. Since the proof is the same in either case, assume $\sigma \in \mathcal{U}$. Let k be an odd integer. Then if $n_k < n \leq n_{k+1}$,

$$\frac{r}{4} \leq \int_{b_n}^{a_n} |\psi_n|^2\, d\lambda \leq \int_{c_{k+1}}^{c_{k-1}} |\psi_n|^2\, d\lambda = d_n.$$

We show that there is a Borel set δ_k in $[c_{k+1}, c_{k-1}]$ so that $\int_{\delta_k} |\psi_n|^2\, d\lambda = \frac{1}{2} d_n$ for $n_k < n \leq n_{k+1}$. Indeed, the map μ defined on the Borel subsets of $[c_{k+1}, c_{k-1}]$ by $\mu(\delta) = (\int_\delta |\psi_{n_k+1}|^2\, d\lambda, \ldots, \int_\delta |\psi_{n_{k+1}}|^2\, d\lambda)$ is a nonatomic vector-valued measure. The range of μ lies in a finite-dimensional vector space; hence, by a theorem of Lyapunov [**16**, p. 114] the range of μ is convex. Since $\mu(\varnothing) = (0, 0, \ldots, 0)$ and $\mu([c_{k+1}, c_{k-1}]) = (d_{n_k+1}, \ldots, d_{n_{k+1}})$, the required set δ_k exists. Let $\delta = \bigcup \{\delta_k\colon k$ an odd integer$\}$ and let $\gamma = [0, 1] - \delta$. Then for any n in σ,

$$\int_\delta |\psi_n|^2\, d\lambda \geq \int_{\delta_k} |\psi_n|^2\, d\lambda = \frac{1}{2} d_n > \frac{r}{8}.$$

Similarly, $\int_\gamma |\psi_n|^2\, d\lambda \geq r/8$. Hence, $r/8 \leq \Lambda_{\mathcal{U}}[\psi_n](P_\delta) \leq 7r/8$, $\Lambda_{\mathcal{U}}[\psi_n]$ is not a homomorphism on $\mathcal{C}$, and the theorem is proved.

8. Ultrafilters with special properties.

(8.1) In this section we study the state $\Lambda_{\mathfrak{U}}[x_n]$ where the free ultrafilter $\mathfrak{U}$ behaves to some degree like a principal ultrafilter. Let $\{\tau_k\}$ be a partition of **N**, let σ be a subset of **N** and let m be an integer. We say that σ is *m-selective for* $\{\tau_k\}$ if the cardinality of $\sigma \cap \tau_k$ is no greater than m for each integer k. We say that σ is *selective for* $\{\tau_k\}$ if there is an integer m so that σ is m-selective for $\{\tau_k\}$. An ultrafilter $\mathfrak{U}$ on **N** is called *rare* if for each partition $\{\tau_k\}$ of **N** into *finite* sets there is a set σ in $\mathfrak{U}$ such that σ is selective for $\{\tau_k\}$. An ultrafilter $\mathfrak{U}$ is called *selective* if for *any* partition $\{\tau_k\}$ of **N**, there is a set σ in $\mathfrak{U}$ which is selective for $\{\tau_k\}$. Note that if $\mathfrak{U}$ is an ultrafilter and σ is an element of $\mathfrak{U}$ which is selective for the partition $\{\tau_k\}$, then there is a set ρ in $\mathfrak{U}$ (in fact $p \subset \sigma$) which is 1-selective for $\{\tau_k\}$. An ultrafilter $\mathfrak{U}$ is called a p-point if for any partition $\{\tau_k\}$ of **N** there is a set σ in $\mathfrak{U}$ such that $\sigma \cap \tau_k$ is finite for each k. Clearly, any principal ultrafilter is rare, selective and a p-point. Also, it is easy to see that an ultrafilter is selective if and only if it is a rare p-point.

Let $\mathfrak{U}$ be a p-point, and let $\{a_n\}$ be a bounded complex sequence. Then there is a subset $\sigma = \{n_j\}$ of **N** so that $\sigma \in \mathfrak{U}$ and $a = \lim_{\mathfrak{U}} a_n = \lim_j a_{n_j}$. For if $\mathfrak{U}$ is principal, the assertion is obvious, and if $\mathfrak{U}$ is a free ultrafilter then let $\tau_\infty = \{n\colon a_n = a\}$, $\tau_1 = \{n\colon |a_n - a| > 1\}$, and $\tau_k = \{n\colon 1/(k-1) > |a_n - a| > 1/k\}$ if $k > 1$. Then $\{\tau_\infty, \tau_1, \ldots\}$ is a partition of **N** such that $\tau_k \notin \mathfrak{U}$ for all integers k. If $\tau_\infty \in \mathfrak{U}$, we may take $\sigma = \tau_\infty$. Otherwise, no set of this partition is in $\mathfrak{U}$ so since $\mathfrak{U}$ is a p-point, there is a set σ in $\mathfrak{U}$ so that $\sigma \cap \tau_k$ is finite for all k. Clearly σ has the desired properties.

Rudin first studied p-points in [15]. Apparently, rare ultrafilters were first introduced by Choquet in [5] and [6] where he also considered selective ultrafilters and p-points. Selective ultrafilters have also been studied by other authors (see [3] for example) and we are not certain where they first appeared in the literature. For more information and further references in this regard, see [14].

Unfortunately, it is not known if a free p-point or a free rare ultrafilter exist, if one assumes only the usual axioms of set theory (Zermelo-Frankel set theory plus the axiom of choice). However, existence proofs have been given using stronger (consistent) set theories. For example, it has been shown using the continuum hypothesis that the sets of p-points, rare ultrafilters, and selective ultrafilters in $\mathfrak{B}\mathbf{N} - \mathbf{N}$ are distinct and that each of these sets form a dense subset of cardinality 2^c [15], [5], [6], [3], [14]. Furthermore, it is conjectured in [14] that it can be shown that p-points exist in $\mathfrak{B}\mathbf{N} - \mathbf{N}$ without resorting to any hypotheses other than the usual axioms of set theory. There is a model of set theory. however in which the only selective ultrafilters are the principal ultrafilters [11], [14]. Hence, in order to construct a selective ultrafilter, one must assume something in addition to the usual axioms of set theory.

(8.2) We continue to consider a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}$. As before, if $\sigma \subset \mathbf{N}$, P_σ is the projection of $\mathcal{H}$ onto $\mathrm{sp}\{e_n: n \in \sigma\}$. In [13] Reid showed that if $\mathcal{U}$ is a rare ultrafilter, then $\Lambda_{\mathcal{U}}[e_n]$ is the unique pure state extension of the homomorphism which $\mathcal{U}$ induces on $\mathcal{D}$ (the diagonal operators in the basis $\{e_n\}$). In the first part of this section we show that the technique which Reid used to prove this fact does not work in all cases.

THEOREM. *If T is an operator and $\{n_j\}$ is a sequence of integers then there is a subsequence $\{r_k\}$ of $\{n_j\}$, a compact operator K and a subset σ of* $\mathbf{N}$ *so that*

$$P_\sigma(T-K)P_\sigma + P_\sigma^\perp(T-K)P_\sigma^\perp \in \mathcal{W}(r_k)$$

where $\mathcal{W}(r_k)$ is the von Neumann algebra defined in (6.1).

PROOF. Let $n_0 = 0$ and let P_j be the projection onto $\mathrm{sp}\{e_n: n_{j-1} < n \leq n_j\}$ for $j = 1, 2, \ldots$. We define the subsequence $\{r_k\}$ of $\{n_j\}$ inductively as follows. Let $r_1 = n_1$ and let $Q_1 = P_1$. If integers $r_1 < r_2 < \cdots < r_k$ and projections $Q_1, Q_2, \ldots, Q_k$ have been chosen, pick r_{k+1} so that $r_{k+1} > r_k$, $r_{k+1} = n_j$ for some j and

$$\|Q_kT(P_1 + P_2 + \cdots + P_j)^\perp\|_{HS}^2 + \|(P_1 + P_2 + \cdots + P_j)^\perp TQ_k\|_{HS}^2 < \left(\tfrac{1}{2}\right)^k,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Let $Q_{k+1} = P_1 + P_2 + \cdots + P_j - (Q_1 + Q_2 + \cdots + Q_k)$. This choice is possible because each Q_k has finite rank and $T(P_1 + P_2 + \cdots + P_j)^\perp$ converges weakly to 0 as $j \to \infty$. Let

$$K = \sum_{k=1}^{\infty} \left(Q_kT(Q_1 + \cdots + Q_{k+1})^\perp + (Q_1 + Q_2 + \cdots + Q_{k+1})^\perp TQ_k\right).$$

Then by the construction K is a Hilbert-Schmidt operator. Let $\sigma = \bigcup_k\{n: r_{2k-1} < n \leq r_{2k}\}$. Then $Q_{2k}KP_\sigma = Q_{2k}T(P_\sigma - Q_{2k})$ so that $Q_{2k}P_\sigma(T-K)P_\sigma = Q_{2k}TQ_{2k}$ and, similarly, $P_\sigma(T-K)P_\sigma Q_{2k} = Q_{2k}TQ_{2k}$. It follows that $P_\sigma(T-K)P_\sigma \in \mathcal{W}(r_k)$. Likewise, $P_\sigma^\perp(T-K)P_\sigma^\perp \in \mathcal{W}(r_k)$.

(8.3) COROLLARY. *If $\mathcal{U}$ is a rare ultrafilter, then for each operator T there is a set τ in $\mathcal{U}$ and a compact operator K so that $P_\tau(T-K)P_\tau \in \mathcal{D}$.*

PROOF. Fix T in $\mathcal{B}(\mathcal{H})$ and let $n_j = j$. By (8.2) $P_\sigma(T-K)P_\sigma + P_\sigma^\perp(T-K)P_\sigma^\perp \in \mathcal{W}(r_k)$ for a sequence $\{r_k\}$ in $\mathbf{N}$, a subset σ of $\mathbf{N}$ and a compact operator K. Then either $\sigma \in \mathcal{U}$ or $\mathbf{N} - \sigma \in \mathcal{U}$. Say $\sigma \in \mathcal{U}$. Let $r_0 = 0$ and for each integer k, let $\tau_k = \{n: r_{k-1} < n \leq r_k\}$. Then $\{\tau_k\}$ is a partition of $\mathbf{N}$ into finite sets so there is a set ρ in $\mathcal{U}$ so that ρ is 1-selective for $\{\tau_k\}$. If $\tau = \rho \cap \sigma$, then $P_\tau(T-K)P_\tau$ is a diagonal operator.

(8.4) Reid's theorem is an easy consequence of (8.3). Indeed, let $\mathcal{U}$ be a rare ultrafilter in $\mathcal{B}\mathbf{N} - \mathbf{N}$ and let $h_{\mathcal{U}}$ be the homomorphism of $\mathcal{D}$ which $\mathcal{U}$ determines. To show that $h_{\mathcal{U}}$ has a unique state extension to $\mathcal{B}(\mathcal{H})$ it suffices to show that $\mathcal{B}(\mathcal{H})$ is $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$ (by (3.2)). So let T be an operator and fix $\varepsilon > 0$. Then by (8.3) $P_\tau(T - K)P_\tau \in \mathcal{D}$ for some τ in $\mathcal{U}$ and some compact operator K. Since K is compact there is a subset σ of τ such that $\tau - \sigma$ is finite and $\|P_\sigma K P_\sigma\| < \varepsilon$. Since $\mathcal{U}$ is a free ultrafilter, $\sigma \in \mathcal{U}$, $\|P_\sigma T P_\sigma - P_\sigma(T - K)P_\sigma\| < \varepsilon$, and $P_\sigma(T - K)P_\sigma = P_\sigma P_\tau(T - K)P_\tau P_\sigma \in \mathcal{D}$. This motivates the following definition.

Definition. Let $\mathcal{Q}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains $\mathcal{D}$ and let $h_{\mathcal{U}}$ be the homomorphism of $\mathcal{D}$ induced by the ultrafilter $\mathcal{U}$. We say that $\mathcal{Q}$ is *strongly $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$* if for each T in $\mathcal{Q}$ and each $\varepsilon > 0$ there is a compact operator K in $\mathcal{Q}$ and a subset σ of $\mathbf{N}$ with σ in $\mathcal{U}$ so that $P_\sigma(T - K)P_\sigma \in \mathcal{D}$.

The foregoing remarks show that if $\mathcal{Q}$ is strongly $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$, then $\mathcal{Q}$ is $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$ and so $h_{\mathcal{U}}$ has a unique state extension to $\mathcal{Q}$. Also, these remarks show that if $\mathcal{U}$ is a rare ultrafilter, then $\mathcal{B}(\mathcal{H})$ is strongly $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$. However, this is not the case for every ultrafilter.

(8.5) Theorem. *There is an ultrafilter $\mathcal{U}$ on* $\mathbf{N}$ *such that $\mathcal{B}(\mathcal{H})$ is not strongly $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$.*

Proof. Assume to the contrary that $\mathcal{B}(\mathcal{H})$ is strongly $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$ for every ultrafilter $\mathcal{U}$. Let T be an operator. Then for each $\mathcal{U}$ in $\mathcal{B}\mathbf{N}$ there is a set $\sigma_{\mathcal{U}}$ and a compact operator $K_{\mathcal{U}}$ so that $P_{\sigma_{\mathcal{U}}}(T - K_{\mathcal{U}})P_{\sigma_{\mathcal{U}}} \in \mathcal{D}$. The collection $\{W(\sigma_{\mathcal{U}})\}$, where $\mathcal{U}$ ranges over all ultrafilters on $\mathbf{N}$, is an open cover. It follows that there is a partition $\{\sigma_1, \ldots, \sigma_n\}$ of $\mathbf{N}$ and a compact operator K so that $P_{\sigma_i}(T - K)P_{\sigma_i} \in \mathcal{D}$ for $1 \leqslant i \leqslant n$. We show that there is a projection P in $\mathcal{B}(\mathcal{H})$ such that for each partition $\{\sigma_1, \ldots, \sigma_n\}$ of $\mathbf{N}$ and each compact operator K, $P_{\sigma_j}(P - K)_{\sigma_i} \notin \mathcal{D}$ for at least one integer i. Let P be a projection such that $\lim_n(Pe_n, e_n) = 0$ and P has infinite rank. (For example, let Q_n be the $n \times n$ matrix each of whose entries is n^{-1} and let $P = \Sigma \oplus Q_n$.) Let $\{\sigma_1, \ldots, \sigma_n\}$ be a partition of $\mathbf{N}$ and let K be a compact operator. Assume that $P_{\sigma_i}(P - K)P_{\sigma_i} \in \mathcal{D}$ for $1 < i < n$. We show that then P is a compact operator, contradicting the fact that the rank of P is infinite and, thus, establishing the theorem. Since K is a compact operator $\lim_n(Ke_n, e_n) = 0$. Let $D_i = P_{\sigma_i}(P - K)P_{\sigma_i}$ for $1 < i < n$. Then $\lim_n(D_ie_n, e_n) = 0$ and D_i is compact for $i = 1, 2, \ldots, n$. Hence, $P_{\sigma_i}P P\sigma_i = P_{\sigma_i}KP_{\sigma_i} + D_i$ is a compact operator for $i = 1, \ldots, n$. Let $R_i = P_{\sigma_i}$ for $i = 1, 2, \ldots, n$. Then $P = \Sigma_{i,j} R_i P R_j$, so $R_i P R_i = R_i P^2 R_i = \Sigma_j R_i P R_j P R_i$ is a compact operator for each i. Since $R_i P R_j P R_i > 0$ for all i and j, $R_i P R_j P R_i$ is a compact operator for all i and j and, hence, R_iPR_j is a compact operator for all i and j $(R_iPR_jPR_i = (R_jPR_i)^*(R_jPR_i))$ so that $P = \sum_{i,j} R_iPR_j$ is a compact operator.

(8.6) REMARK. It can be shown (assuming the continuum hypothesis) that there is a free ultrafilter $\mathcal{U}$ which is not rare, but such that $\mathcal{B}(\mathcal{H})$ is strongly $\mathcal{D}$-compressible modulo $h_{\mathcal{U}}$. The proof is fairly long, however, so we merely present an outline. Fix a partition $\{\tau_k\}$ of $\mathbf{N}$ into finite, but unbounded intervals. If T is an operator with 0 diagonal and τ is a subset of N which is not selective for $\{\tau_k\}$, then there is a subset σ of τ which is not selective for $\{\tau_k\}$ but so that $P_\sigma TP_\sigma$ is a compact operator. Well-order all operators and all subsets of $\mathbf{N}$ as $\{T_\alpha\}$ and $\{\rho_\alpha\}$, respectively, where α runs through the countable ordinals (continuum hypothesis). A collection σ_α of subsets of $\mathbf{N}$ is constructed by transfinite induction. At the αth step of the construction we assume that for all ordinals $\beta < \alpha$ sets σ_β have been selected so that:

(i) σ_β is not selective for $\{\tau_k\}$.

(ii) Either $\sigma_\beta \subseteq \rho_\beta$ or $\sigma_\beta \subseteq \mathbf{N} - \rho_\beta$.

(iii) The sets $\{W(\sigma_\gamma)\}$ are decreasing for all $\gamma < \beta$.

(iv) $P_{\sigma_\beta}(T_\beta - K_\beta)P_{\sigma_\beta} \in \mathcal{D}$ for some compact operator K_β.

Since there are only a countable number of β's less than α, by (i) and (iii) it is possible to construct an infinite set τ_α such that all but a finite number of elements of τ_α belong to each σ_β, τ_α is not selective for $\{\tau_k\}$ and either $\tau_\alpha \subset \rho_\alpha$ or $\tau_\alpha \subset \mathbf{N} - \rho_\alpha$. Let D_α be the diagonal of T_α. Then there is a subset σ_α of τ_α so that $P_{\sigma_\alpha}(T_\alpha - D_\alpha)P_{\sigma_\alpha} = K_\alpha$, a compact operator and σ_α is not selective for $\{\tau_k\}$.

(8.7) We say that a state f on $\mathcal{B}(\mathcal{H})$ has the *continuous restriction property* if there is a maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ which is unitarily equivalent to $\mathcal{C}$ (as defined in §7) and such that f is a homomorphism on $\mathcal{B}$. We say that f has the *discrete restriction property* if there is a maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ which is unitarily equivalent to $\mathcal{D}$ and such that f is a homomorphism on $\mathcal{B}$. Recall that any maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$ is unitarily equivalent to $\mathcal{D}$, $\mathcal{C}$, $\mathcal{C} \oplus \mathcal{D}$, or $\mathcal{C} \oplus \mathcal{T}$ where $\mathcal{T}$ is the finite dimensional analogue of $\mathcal{D}$. It follows that if a state f on $\mathcal{B}(\mathcal{H})$ is a homomorphism on a maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$, then f has either the continuous restriction property or the discrete restriction property. By (7.3) there are pure states on $\mathcal{B}(\mathcal{H})$ which have both the continuous restriction property and the discrete restriction property. We now show that pure states given by selective ultrafilters have the discrete restriction property but do not have the continuous restriction property.

(8.8) THEOREM. *If $\mathcal{U}$ is a p-point and $\{x_n\}$ is a sequence of unit vectors in $\mathcal{H}$, then $\Lambda_{\mathcal{U}}[x_n]$ does not have the continuous restriction property.*

PROOF. Suppose that $\Lambda_{\mathcal{U}}[x_n]$ is a homomorphism on a maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ and that $S\mathcal{B}S^* = \mathcal{C}$ for some unitary transformation S of $\mathcal{H}$ onto $L^2(0, 1)$. Let $\psi_n = Sx_n$ for each n. Then $\Lambda_{\mathfrak{U}}[\psi_n]$ agrees with a homomorphism h on $\mathcal{C}$, where h is in the fiber over a for some a in $[0, 1]$. Assume that $a < 1$ and $h(P_{[a, 1]}) = 1$. The proof in the other cases is the same. Then passing to a subsequence if necessary, we may assume that $\int_a^1 |\psi_n|^2 \, d\lambda > 3/4$, for all n. Choose a_n so that $\int_a^{a_n} |\psi_n|^2 \, d\lambda = \frac{1}{2}$. Since $\mathfrak{U}$ is a p-point, as we noted in (8.1) there is a subset $\sigma = \{n_j\}$ of $\mathbf{N}$ with σ in $\mathfrak{U}$ such that $\lim_{\mathfrak{U}} a_n = \lim_j a_{n_j} = a'$. If $a < a'$ then $\Lambda_{\mathfrak{U}}[\psi_n](P_{[a, a']}) = \frac{1}{2}$ and $\Lambda_{\mathfrak{U}}[\psi_n]$ is not a homomorphism on $\mathcal{C}$. So suppose $a = a'$ and let $a'_n = a_n$ if $n \in \sigma$, $a'_n = a$ if $n \notin \sigma$. Then $\lim_n a'_n = a$ and $\lim_{\mathfrak{U}} \int_a^{a'_n} |\psi_n|^2 \, d\lambda = \frac{1}{2}$ so that $\Lambda_{\mathfrak{U}}[\psi_n]$ is not a homomorphism on $\mathcal{C}$ by (7.5). Hence, $\mathcal{B}$ is not unitarily equivalent to $\mathcal{C}$ and the theorem is proved.

(8.9) COROLLARY. *If $\mathfrak{U}$ is a selective ultrafilter, then $\Lambda_{\mathfrak{U}}[e_n]$ is a pure state with the discrete restriction property which does not have the continuous restriction property.*

PROOF. As we noted in (8.1), if $\mathfrak{U}$ is selective, then $\mathfrak{U}$ is a rare p-point. Hence, $\Lambda_{\mathfrak{U}}[e_n]$ is a pure state on $\mathcal{B}(\mathcal{H})$ (by Reid's theorem) which clearly has the discrete restriction property, but since $\mathfrak{U}$ is a p-point $\Lambda_{\Omega}[e_n]$ does not have the continuous restriction property by (8.8).

(8.10) REMARK. In fact if $\mathfrak{U}$ is a selective ultrafilter and $\{x_n\}$ is a sequence of unit vectors such that $x_n \rightharpoonup O(\mathfrak{U})$, then it can be shown that there is an orthonormal basis $\{f_n\}$ for $\mathcal{H}$ such that $\Lambda_{\mathfrak{U}}[x_n] = \Lambda_{\mathfrak{U}}[f_n]$. Hence, $\Lambda_{\mathfrak{U}}[x_n]$ is a pure state with the discrete restriction property which does not have the continuous restriction property.

(8.11) REMARK. Alexander Kechris has pointed out that the statement: "$\mathcal{B}(\mathcal{H})$ is orthogonally $\mathfrak{D}$-compressible" is what is known as a Π_2^1 statement. (Roughly speaking, a Π_2^1 statement is one of the form: "For every real number there exists a real number such that A", where A does not involve any further quantification over the reals.) By the Shoenfield absoluteness lemma **[19]**, if such a statement can be proved (or disproved) assuming the continuum hypothesis, then it can be proved (or disproved) without this assumption.

REFERENCES

1. J. Anderson, *Derivations, commutators and the essential numerical range*, Ph. D. thesis, Indiana Univ., Bloomington, Ind., 1971.

2. S. K. Berberian, *Equivalence of projections*, Proc. Amer. Math. Soc. **33** (1972), 485–489.

3. D. Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970), 1–24.

4. A. Brown and C. Pearcy, *Structure of commutators of operators on Hilbert space*, Ann. of Math. (2) **82** (1965), 112–127.

5. G. Choquet, *Construction d'ultrafiltres sur* **N**, Bull. Sci. Math. (2) **92** (1968), 41–48.

6. ______, *Deux classes remarquables d'ultrafiltres sur* **N**, Bull. Sci. Math. (2) **92** (1968), 143–153.

7. J. Dixmier, *Les C^*-algebres et leur représentations*, Gauthier-Villars, Paris, 1964.

8. J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras*, Ann. of Math. (2) **72** (1960), 216–244.

9. R. V. Kadison, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276.

10. R. V. Kadison and I. M. Singer, *Extensions of pure states*, Amer. J. Math. **81** (1959), 383–400.

11. K. Kunen, *On the compactification of the integers*, Notices Amer. Math. Soc. **17** (1970), 299.

12. J. T. Marti, *Introduction to the theory of bases*, Springer Tracts in Natural Philosophy, vol. 18, Springer-Verlag, New York, 1969.

13. G. A. Reid, *On the Calkin representations*, Proc. London Math. Soc. (3) **23** (1971), 547–564.

14. M. E. Rudin, *Interaction of set theory and general topology*, CBMS Regional Conference, August 12–16, 1974, Laramie, Wyoming.

15. W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419, 633.

16. ______, *Functional analysis*, McGraw-Hill, New York, 1973.

17. ______, *Real and complex analysis*, 2nd ed., McGraw-Hill, New York, 1974.

18. ______, *Sums of ideals*, Notices Amer. Math. Soc. **21** (1974), A-189.

19. J. R. Shoenfield, *The problem of predicativity*, Essays on the Foundations of Mathematics, Magnes Press, Hebrew Univ., Jerusalem, 1961, pp. 132–139.

20. I. M. Wils, *Stone-Čech compactifications and representations of operator algebras*, Ph. D. thesis, Catholic Univ. of Nijemgin, 1968.

DEPARTMENT OF MATHEMATICS CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125

Current address: Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802